Height functions associated with closed subschemes

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Abstract

This is an expository note on height functions associated with closed subschemes. This manuscript contains nothing new.

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1 Introduction

This is an expository note on height function associated with closed subschemes with detailed proof. Height functions associated with closed subschemes are introduced by Silverman in [7]. They are generalization of usual height functions associated to Cartier divisors. Standard references for height functions associated with Cartier divisors are [2, 3, 4].

In most literatures, the base schemes on which we define height functions are assumed to be irreducible and reduced. In this note, we do not assume these properties.

2 Notation

Let k be a field.

(1) A projective scheme over k is a scheme over k which has a closed immersion to the projective space $\mathbb{P}^n_k$ for some $n \in \mathbb{Z}_{>0}$ over k.

(2) A quasi-projective scheme over k is a scheme over k which has an open immersion to a projective scheme over k.

(3) An algebraic scheme over k is a separated scheme of finite type over k.

(4) Let X be a scheme over k and $k \subset k'$ be a field extension. The base change $X \times_{\operatorname{Spec} k} \operatorname{Spec} k'$ is denoted by $X_{k'}$. For an "object" A on X, we sometimes use the notation $A_{k'}$ to express the base change of A to k' without mentioning to the definition of the base change if the meaning is clear.

(5) For a Noetherian scheme X, the set of associated point of X is denoted by $\operatorname{Ass}(X)$.

3 Absolute Values

3.1 Absolute values

In this section, we recall basic facts on absolute values that we use to define height function later. A good reference for absolute values is [6, Chapter 9].

Definition 3.1. Let K be a field. An absolute value on K is a map

$$|\ |_v \colon K \longrightarrow \mathbb{R}_{\geqslant 0}; x \mapsto |x|_v$$

which satisfies the following properties:

(1) $|xy|_v = |x|_v|y|_v$ for all $x, y \in K$;

(2) $|x + y|_v \leqslant |x|_v + |y|_v$ for all $x, y \in K$;

(3) for $x \in K$, $|x|_v = 0$ if and only if $x = 0$.

The absolute value $|\ |_v$ is called non-archimedean if the following holds instead of (2):

$$|x + y|_v \leqslant \max\{|x|_v, |y|_v\} \quad \text{for all } x, y \in K.$$

An absolute value which is not non-archimedean is called archimedean. Two absolute values are said to be equivalent of they define the same topology on K.

Remark 3.2. An absolute value $|\ |_v$ is non-archimedean if and only if $|n{\cdot}1|_v \leqslant 1$ for all $n \in \mathbb{Z}$ (the 1 is the identity element of our field K).

We use the subscript v to distinguish several absolute values. We sometimes refer an absolute value as v instead of $|\ |_v$ for simplicity. In this note, even if we write simply v, this does not mean the place.

Definition 3.3. Let K be a field. Let $|\ |_v$ be a non-trivial absolute value on K. We say $|\ |_v$ is well-behaved if for any finite field extension $K \subset L$, we have

$$[L : K] = \sum_{w|v} [L_w : K_v]$$

where the sum runs over all absolute values w on L which extend v. (The notation $w|v$ means the restriction of w on K is v.) Here L_w and K_v are the completion of L and K with respect to w, v respectively.

Remark 3.4. In the setting of the above definition, the inequality $[L : K] \geqslant \sum_{w|v}[L_w : K_v]$ is always true.

Remark 3.5. If K is complete with respect to v, then v is well-behaved because there is exactly one extension of v to L and L is complete with respect to that absolute value.

Remark 3.6. Let K be a field of characteristic zero. Then every non-trivial absolute value on K is well-behaved.

Proof. Let v be a non-trivial absolute value on K. Let L be a finite extension of K. Then the extensions of v to L correspond to the prime ideals of $L \otimes_K K_v$. Since K has characteristic zero, we have

$$L \otimes_K K_v \simeq L_{w_1} \times \cdots \times L_{w_r}$$

where $w_1, \ldots, w_r$ are the extensions of v to L. Thus we get

$$[L : K] = \dim_{K_v} L \otimes_K K_v = \sum_{i=1}^{r} [L_{w_i} : K_v].$$

□

Lemma 3.7. *Let K be a field and v a well-behaved absolute value on K. Then for any finite extension $K \subset L$ and any extension w of v to L, w is also a well-behaved absolute value.*

Proof. Let L' be a finite extension L. Then we get

$$\begin{aligned}
[L' : K] &= \sum_{\substack{u \mid v \\ u \text{ ab. val. on } L'}} [L'_u : K_v] \qquad \text{since } v \text{ is well-behaved} \\
&= \sum_{\substack{w \mid v \\ w \text{ ab. val.} \\ \text{on } L}} \sum_{\substack{u \mid w \\ u \text{ ab. val.} \\ \text{on } L'}} [L'_u : L_w][L_w : K_v] \\
&\leqslant [L' : L] \sum_{\substack{w \mid v \\ w \text{ ab. val.} \\ \text{on } L}} [L_w : K_v] = [L' : L][L : K] = [L' : K].
\end{aligned}$$

Therefore, the inequality is actually equality and we are done.

□

Lemma 3.8. *Let K be a field and v a well-behaved absolute value on K. Let $K \subset L$ be a finite extension. Then for any $\alpha \in L$, we have*

$$\prod_{\substack{w \mid v, \\ w \text{ ab. val. on } L}} |\alpha|_w^{[L_w : K_v]} = |N_{L/K}(\alpha)|_v.$$

Here $N_{L/K}$ is the norm associated to the field extension.

Proof. We have

$$N_{L/K}(\alpha) = \det_K(L \xrightarrow{\alpha} L) = \det_{K_v}(L \otimes_K K_v \xrightarrow{\alpha\otimes 1} L \otimes_K K_v)$$

where $\det_K$ (resp. $\det_{K_v}$) stand for determinant as K (resp. K_v) linear maps. Since v is well-behaved, we have

$$L \otimes_K K_v \simeq \prod_{w|v} L_w.$$

Thus we get

$$N_{L/K}(\alpha) = \prod_{w|v} \det_{K_v}(L_w \xrightarrow{\alpha} L_w) = \prod_{w|v} N_{L_w/K_v}(\alpha).$$

This shows

$$|N_{L/K}(\alpha)|_v = \prod_{w|v} |N_{L_w/K_v}(\alpha)|_v = \prod_{w|v} |\alpha|_w^{[L_w:K_v]}.$$

□

3.2 Proper set of absolute values

Definition 3.9. Let K be a field and $|\ |_v$ be an absolute value on K. We say $|\ |_v$ is proper if $|\ |_v$ is non-trivial, well-behaved, and if $\operatorname{char} K = 0$, then the restriction of $|\ |_v$ to $\mathbb{Q}$ is equivalent to either trivial absolute value, p-adic absolute value, or the usual absolute value[1].

Definition 3.10. Let K be a field. A non-empty set M_K of absolute values on K is said to be proper if

(1) all element of M_K is proper;

(2) for all $|\ |_v, |\ |_w \in M_K$, if $|\ |_v \neq |\ |_w$, then $|\ |_v$ and $|\ |_w$ are not equivalent;

(3) for all $x \in K \backslash \{0\}$, $\sharp\{v \in M_K \mid |x|_v \neq 1\} < \infty$.

Remark 3.11. A proper set of absolute value M_K contains only finitely many archimedean absolute values. This is because the property (3) and archimedean absolute values restrict to the usual absolute value on $\mathbb{Q}$. We denote the set of archimedean absolute values in M_K by M_K^∞. The set of non-archimedean absolute values is denoted by M_K^{fin}.

[1]This is automatically true.

Remark 3.12. Since finite fields do not have non-trivial absolute values, K is infinite if it admits a proper set of absolute values.

Definition 3.13. Let K be a field and M_K be a proper set of absolute values on K. For a field extension $K \subset L$, we set

$$M_K(L) = M(L) = \left\{ |\ |_w \;\middle|\; \begin{array}{c} |\ |_w \text{ is an absolute value on } L \\ \text{whose restriction to } K \text{ is an element of } M_K \end{array} \right\}.$$

Strictly speaking, this set depends on M_K and the embedding $K \to L$. But we usually fix K, M_K, and an algebraic closure $\overline{K}$ of K and work inside $\overline{K}$, so there would not be any confusion.

Lemma 3.14. *Let K be a field and M_K be a proper set of absolute values on K. Let $K \subset L$ be a finite extension. Then $M(L)$ is a proper set of absolute values on L.*

Proof. The only non-trivial thing is property (3) in Definition 3.10. Let $\alpha \in L \backslash \{0\}$. We will show that $\sharp\{w \in M(L) \mid |\alpha|_w \neq 1\} < \infty$. Let $f(X) = X^n + a_{n-1}X^{n-1} + \cdots + a_0$ be the minimal monic polynomial of α over K. Since M_K is proper, there is a finite set $S \subset M_K$ containing M_K^∞ such that

$$\begin{aligned} &|a_i|_v = 1 \text{ or } 0 \text{ for } i = 0, \ldots, n-1 \\ &|N_{L/K}(\alpha)|_v = 1 \end{aligned}$$

for all $v \in M_K \backslash S$. For any $v \in M_K \backslash S$, let $w \in M(L)$ be such that $w|v$. Then α (as an element of L_w) is integral over $\mathcal{O}_v = \{x \in K_v \mid |x|_v \leqslant 1\}$. Thus $|N_{L_w/K_v}(\alpha)|_v \leqslant 1$. By the proof of Lemma 3.8 and $|N_{L/K}(\alpha)|_v = 1$, we get $|N_{L_w/K_v}(\alpha)|_v = 1$ for all $w \in M(L)$ such that $w|v$. Thus $|\alpha|_w = |N_{L_w/K_v}(\alpha)|_v^{1/[L_w:K_v]} = 1$ for all $w \in M(L)$ such that $w|_K \in M_K \backslash S$. $\square$

4 Local heights

In this section, we define local height functions associated to closed subschemes. We will do this in several steps.

4.1 Presentation of closed subschemes

In this section, we work over an infinite field k.

Lemma 4.1. *Let $U \subset \mathbb{A}_k^N$ be a non-empty Zariski open subset. Then $U(k) \neq \varnothing$.*

Proof. (sketch).

We may assume $U = D(f)(= \mathbb{A}_k^N \backslash (f = 0))$, where $f \in k[x_1, \ldots, x_N]$ is a non zero polynomial. It is enough to show that for any non zero polynomial $f \in k[x_1, \ldots, x_N]$, there is a point $a \in k^N$ such that $f(a) \neq 0$. This can be shown by induction on N. For $N = 1$ case and proceeding induction, we need the assumption that k is infinite. □

For an effective Cartier divisor D on X, let $\mathcal{O}_X(D) = \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X)$ where $\mathcal{I}_D$ is the ideal sheaf of D. The global section of $\mathcal{O}_X(D)$ corresponding the inclusion $\mathcal{I}_D \to \mathcal{O}_X$ is denoted by s_D.

Lemma 4.2. *Let X be a quasi-projective scheme over k.*

(1) *For any effective Cartier divisor D on X, there are globally generated invertible $\mathcal{O}_X$-modules $\mathcal{L}, \mathcal{M}$ such that $\mathcal{O}_X(D) \simeq \mathcal{L} \otimes \mathcal{M}^{-1}$.*

(2) *Let $\mathcal{L}$ be a globally generated invertible $\mathcal{O}_X$-module on X and $x_1, \ldots x_n \in X$ be scheme points. Then there are generating global sections $s_0, \ldots, s_n$ of $\mathcal{L}$ such that non of them vanish at any of x_i.*

(3) *Let $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Let $x_1, \ldots x_n \in X \backslash Y$ be scheme points. Then there are effective Cartier divisors $D_1, \ldots, D_r$ on X such that*

$$Y = D_1 \cap \cdots \cap D_r \qquad \text{as closed subschemes of } X;$$
$$x_1, \ldots, x_n \notin D_i \qquad \text{for } i = 1, \ldots, r.$$

Proof. (1) Since X is quasi-projective, there is an ample invertible $\mathcal{O}_X$-module $\mathcal{O}_X(1)$. We will write $\mathcal{O}_X(m) = \mathcal{O}_X(1)^{\otimes m}$. For large enough $m > 0$, $\mathcal{O}_X(m)$ and $\mathcal{O}_X(D) \otimes \mathcal{O}_X(m)$ are globally generated, and we are done.

(2) Take a morphism to a projective space $\varphi \colon X \longrightarrow \mathbb{P}_k^N$ defined by a generating global sections of $\mathcal{L}$. Replace X, $\mathcal{L}$, and $x_1, \ldots x_n$ with $\mathbb{P}_k^N$, $\mathcal{O}(1)$, and $\varphi(x_1), \ldots, \varphi(x_n)$, we may assume $X = \mathbb{P}_k^N$ and $\mathcal{L} = \mathcal{O}(1)$. Non vanishing at $x_1, \ldots, x_n$ is a Zariski open condition on $H^0(\mathcal{O}(1)) = k^{N+1}$: let $U \subset H^0(\mathcal{O}(1))$ be the open set. Generating $\mathcal{O}(1)$ is a Zariski open condition for $N+1$-tuples of elements of $H^0(\mathcal{O}(1))$: let $V \subset H^0(\mathcal{O}(1))^{N+1}$ be the open set. Then by Lemma 4.1, $U^{N+1} \cap V \neq \varnothing$ and we are done.

(3) It follows from the following claim.

Claim 4.3. *Let X be a projective scheme and $Y \subset X$ a closed subscheme. Let $x_1, \ldots, x_n \in X \backslash Y$ be scheme points. Then there are locally principal closed subschemes $D_1, \ldots, D_r$ of X such that $Y = D_1 \cap \cdots \cap D_r$ as schemes and $x_i \notin D_j$ for all i, j.*

Proof. Let $\mathcal{I}$ be the ideal of $Y \subset X$. By replacing some specialization points, we may assume x_i are closed points of X. Let $\mathfrak{m}_{x_i}$ be the ideal of the closed points x_i. Let $\mathcal{O}_X(1)$ be an ample $\mathcal{O}_X$-module.

Let us consider the following exact sequence:

$$0 \longrightarrow \mathfrak{m}_{x_1} \cdots \mathfrak{m}_{x_n} \longrightarrow \mathcal{O}_X \longrightarrow \bigoplus_{i=1}^{n} k(x_i) \longrightarrow 0.$$

Since $Y \cap \{x_1, \ldots, x_n\} = \varnothing$, we can tensor $\mathcal{I}$ preserving exactness:

$$0 \longrightarrow \mathfrak{m}_{x_1} \cdots \mathfrak{m}_{x_n} \otimes \mathcal{I} \longrightarrow \mathcal{I} \longrightarrow \bigoplus_{i=1}^{n} k(x_i) \longrightarrow 0.$$

Let us take $m > 0$ so that $\mathcal{I} \otimes \mathcal{O}_X(m)$ is globally generated and $H^1(X, \mathfrak{m}_{x_1} \cdots \mathfrak{m}_{x_n} \otimes \mathcal{I} \otimes \mathcal{O}_X(m)) = 0$. Then we get a surjection:

$$H^0(X, \mathcal{I} \otimes \mathcal{O}_X(m)) \longrightarrow \bigoplus_{i=1}^{n} k(x_i).$$

Take k-bases of $k(x_i)$ and fix a k-vector isomorphism $\bigoplus_{i=1}^{n} k(x_i) \simeq k^d$. Write the surjective k-linear map defined by the above map $\pi\colon H^0(X, \mathcal{I} \otimes \mathcal{O}_X(m)) \longrightarrow k^d$. Since$k$ is infinite, by Lemma 4.1, there is a k-basis $e_1, \ldots, e_d$ of k^d whose all coordinates are not zero. Let $s_1, \ldots, s_d \in H^0(X, \mathcal{I} \otimes \mathcal{O}_X(m))$ be lifts of e_i: $\pi(s_i) = e_i$. Extend this to generating sections $s_1, \ldots, s_d, s'_{d+1}, \ldots, s'_r$. Since $e_1, \ldots, e_d$ is a k-basis of k^d, there are $a_{ij} \in k$ such that all coordinates of

$$\pi(s'_i + \sum_{j=1}^{d} a_{ij} s_j) \quad i = d+1, \ldots, r$$

are not zero. Set $s_i = s'_i + \sum_{j=1}^{d} a_{ij} s_j$ for $i = d+1, \ldots, r$. These $s_1, \ldots, s_r$ as elements of $H^0(X, \mathcal{O}_X(m))$ define locally principal closed subschemes $D_1, \ldots, D_r$ of X. By the construction, $Y = D_1 \cap \cdots \cap D_r$ and $x_i \notin D_j$. ■

□

Definition 4.4 (Presentations of closed subschemes). Let X be a quasi-projective scheme over k.

(1) Let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. A global section $s \in H^0(X, \mathcal{L})$ is called a regular global section if the homomorphism $\mathcal{O}_X \longrightarrow \mathcal{L}$ defined by s is injective. Note that this is equivalent to say that s does not vanish at any of associated points of X.

(2) Let D be an effective Cartier divisor on X. A presentation of D is

$$\mathcal{D} = (s_D; \mathcal{L}, s_0, \ldots, s_n; \mathcal{M}, t_0, \ldots, t_m; \psi)$$

where $\mathcal{L}$, $\mathcal{M}$ are globally generated invertible $\mathcal{O}_X$-modules, $\psi\colon \mathcal{L} \otimes \mathcal{M}^{-1} \to \mathcal{O}_X(D)$ is an isomorphism, and $s_0, \ldots, s_n \in H^0(X, \mathcal{L})$ and $t_0, \ldots, t_m \in H^0(X, \mathcal{M})$ are regular global sections which generate $\mathcal{L}$ and $\mathcal{M}$. We usually omit ψ and write

$$\mathcal{D} = (s_D; \mathcal{L}, s_0, \ldots, s_n; \mathcal{M}, t_0, \ldots, t_m)$$

for simplicity.

(3) Let $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. A presentation of Y is

$$\mathcal{Y} = (Y; \mathcal{D}_1, \ldots, \mathcal{D}_r)$$

where $\mathcal{D}_i$ are presentations of effective Cartier divisors D_i on X such that $Y = D_1 \cap \cdots, \cap D_r$ as closed subschemes.

Remark 4.5. By Lemma 4.2, those presentations are always exist.

Definition 4.6. Let X be a quasi-projective scheme over k. Let D, E be two effective Cartier divisors on X and

$$\begin{aligned}\mathcal{D} &= (s_D; \mathcal{L}, s_0, \ldots, s_n; \mathcal{M}, t_0, \ldots, t_m; \psi);\\ \mathcal{E} &= (s_E; \mathcal{L}', s'_0, \ldots, s'_{n'}; \mathcal{M}', t'_0, \ldots, t'_{m'}; \psi')\end{aligned}$$

presentations of them.

(1) We define the sum of $\mathcal{D}$ and $\mathcal{E}$ as

$$\mathcal{D} + \mathcal{E} = \Big(s_{D+E}; \mathcal{L} \otimes \mathcal{L}', \{s_i \otimes s'_j\}_{\substack{0 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant n'}}; \mathcal{M} \otimes \mathcal{M}', \{t_i \otimes t'_j\}_{\substack{0 \leqslant i \leqslant m \\ 0 \leqslant j \leqslant m'}}; \psi \otimes \psi'\Big).$$

Note that this is a presentation of $D + E$.

(2) Let $f\colon X' \longrightarrow X$ be a morphism over k where X' is a quasi-projective scheme. We say $f^*\mathcal{D}$ is well-defined if $f(\mathrm{Ass}(X')) \cap D = \varnothing$ and $s_0, \ldots, s_n$, $t_0, \ldots, t_m$ do not vanish at any points of $f(\mathrm{Ass}(X'))$. In this case, the scheme theoretic inverse image $f^{-1}(D)$ is an effective Cartier divisor on X', which is denoted by f^*D, and we define

$$f^*\mathcal{D} = (s_{f^*D}; f^*\mathcal{L}, f^*s_0, \ldots, f^*s_n; f^*\mathcal{M}, f^*t_0, \ldots, f^*t_m; f^*\psi).$$

This is a presentation of f^*D.

Definition 4.7. Let X be a quasi-projective scheme over k. Let $Y, W \subset X$ be closed subschemes such that $Y \cap \mathrm{Ass}(X) = W \cap \mathrm{Ass}(X) = \varnothing$. Let $\mathcal{I}_Y, \mathcal{I}_W$ be the ideals of Y, W. The closed subschemes defined by $\mathcal{I}_Y + \mathcal{I}_W$ and $\mathcal{I}_Y\mathcal{I}_W$ are denoted by $Y \cap W$ and $Y + W$ respectively. Note that $(Y \cap W) \cap \mathrm{Ass}(X) = \varnothing$ and $(Y + W) \cap \mathrm{Ass}(X) = \varnothing$.

Let

$$\mathcal{Y} = (Y; \mathcal{D}_1, \ldots, \mathcal{D}_r);$$
$$\mathcal{W} = (W; \mathcal{E}_1, \ldots, \mathcal{E}_s)$$

be presentations of Y, W.

(1) We define

$$\mathcal{Y} \cap \mathcal{W} = (Y \cap W; \mathcal{D}_1, \ldots, \mathcal{D}_r, \mathcal{E}_1, \ldots, \mathcal{E}_s).$$

This is a presentation of $Y \cap W$.

(2) We define

$$\mathcal{Y} + \mathcal{W} = (Y + W; \{\mathcal{D}_i + \mathcal{E}_j\}_{1 \leqslant i \leqslant r, 1 \leqslant j \leqslant s}).$$

This is a presentation of $Y + W$.

(3) Let $f\colon X' \longrightarrow X$ be a morphism over k where X' is a quasi-projective scheme. Suppose $f^*\mathcal{D}_i$ are well-defined for all $i = 1, \ldots, r$. (In this case, we say $f^*\mathcal{Y}$ is well-defined.) Then $f^{-1}(Y) \cap \mathrm{Ass}(X') = \varnothing$. We define

$$f^*\mathcal{Y} = (f^{-1}(Y); f^*\mathcal{D}_1, \ldots, f^*\mathcal{D}_r).$$

This is a presentation of $f^{-1}(Y)$.

Remark 4.8. Let $f\colon X' \longrightarrow X$ be a morphism over k between quasi-projective schemes over k. Let $Y \subset X$ be a closed subscheme such that $Y \cap (\mathrm{Ass}(X) \cup f(\mathrm{Ass}(X'))) = \varnothing$. Then by Lemma 4.2, there is a presentation $\mathcal{Y}$ of Y such that $f^*\mathcal{Y}$ is well-defined.

Lemma 4.9. *Let $k \subset k'$ be a field extension. Let X be a quasi-projective scheme over k and $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Let $\mathcal{Y} = (Y; \mathcal{D}_1, \ldots, \mathcal{D}_r)$ be a presentation of Y where*

$$\mathcal{D}_i = (s_{D_i}; \mathcal{L}^{(i)}, s_0^{(i)}, \ldots s_{n_i}^{(i)}; \mathcal{M}^{(i)}, t_0^{(i)}, \ldots, t_{m_i}^{(i)}).$$

Then the base change $Y_{k'} \subset X_{k'}$ also satisfies $Y_{k'} \cap \mathrm{Ass}(X_{k'}) = \varnothing$ and

$$\mathcal{Y}_{k'} := (Y_{k'}; (\mathcal{D}_1)_{k'}, \ldots (\mathcal{D}_r)_{k'})$$

is a presentation of $Y_{k'}$ where

$$(\mathcal{D}_i)_{k'} = (s_{(D_i)_{k'}}; \mathcal{L}_{k'}^{(i)}, (s_0^{(i)})_{k'}, \ldots (s_{n_i}^{(i)})_{k'}; \mathcal{M}_{k'}^{(i)}, (t_0^{(i)})_{k'}, \ldots, (t_{m_i}^{(i)})_{k'}).$$

Proof. Since $X_{k'} \longrightarrow X$ is flat, $\mathrm{Ass}(X')$ is mapped to $\mathrm{Ass}(X)$. The rest is obvious. $\square$

Construction 4.10 (Meromorphic functions associated to presentations)**.**

Let X be an algebraic scheme and $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module. For a regular section $s \in H^0(X, \mathcal{L})$, we define a section $s^{-1} \in H^0(U, \mathcal{L}^{-1})$ where $U = X \backslash \operatorname{div}(s)$ as follows. Since

$$s\colon \mathcal{O}_U \longrightarrow \mathcal{L}|_U$$

is an isomorphism, the homomorphism

$$s^\vee : \mathcal{L}^{-1} := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{L}, \mathcal{O}_X) \longrightarrow \mathcal{H}om_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{O}_X)$$

is also an isomorphism on U. We define $s^{-1} = (s^\vee)^{-1}(\mathrm{id}_{\mathcal{O}_U}) \in H^0(U, \mathcal{L}^{-1})$. Note that U is dense open in X since s is a regular section.

Let X be a quasi-projective scheme over k and D be an effective Cartier divisor on X. Take a presentation of D:

$$\mathcal{D} = (s_D; \mathcal{L}, s_0, \ldots, s_n; \mathcal{M}, t_0, \ldots, t_m; \psi).$$

Then we have the following isomorphism:

$$\Psi\colon \mathcal{L} \otimes \mathcal{M}^{-1} \otimes \mathcal{O}_X(D)^{-1} \xrightarrow{\psi \otimes \mathrm{id}} \mathcal{O}_X(D) \otimes \mathcal{O}_X(D)^{-1} \simeq \mathcal{O}_X$$

where the last isomorphism is the canonical isomorphism. We write

$$\frac{s_i}{s_D t_j} := \Psi(s_i \otimes t_j^{-1} \otimes s_D^{-1}) \in H^0(U, \mathcal{O}_X)$$

where $U = X \backslash (D \cup \operatorname{div}(t_j))$ is a dense open of X. Let $V = \operatorname{Spec} A \subset U$ be an open subset on which $\mathcal{L}$ and $\mathcal{M}$ are trivial. Fix isomorphisms $\mathcal{L}|_V \simeq \mathcal{O}_V$ and $\mathcal{M}|_V \simeq \mathcal{O}_V$. By these isomorphisms and ψ, we get isomorphisms $\mathcal{O}_X(D)|_V \simeq \mathcal{L}|_V \otimes \mathcal{M}|_V^{-1} \simeq \mathcal{O}_V$. Let $f, g, f_D \in A$ be elements corresponding to $s_i|_V, t_j|_V, s_D|_V$ via these isomorphisms. Then $t_j^{-1}|_V$ and $s_D^{-1}|_V$ correspond to g^{-1} and f_D^{-1} and

$$\left.\frac{s_i}{s_D t_j}\right|_V \quad \text{corresponds to} \quad \frac{f}{f_D g}.$$

4.2 Bounded subsets

In this section, we fix a field K and a proper set of absolute values M_K on K. We fix an algebraic closure $K \subset \overline{K}$. Let $M = M(\overline{K}) = \{v \mid v \text{ is an absolute value on } \overline{K} \text{ such that } v|_K \in M_K\}$.

For absolute value v on any field, let $\epsilon(v) = 1$ if v is archimedean and $\epsilon(v) = 0$ if v is non-archimedean.

Definition 4.11. Let $K \subset L \subset \overline{K}$ be an intermediate field such that $[L : K] < \infty$. Let X be an algebraic scheme over L.

(1) Let $U \subset X$ be an open affine subset. Let $v \in M(\overline{K})$. A subset $B \subset U(\overline{K})$ is said to be bounded with respect to U and v if

$$\sup_{x \in B}\{|f(x)|_v\} < \infty \quad \text{for all } f \in \mathcal{O}(U).$$

(2) Let $U \subset X$ be an open affine subset. An M_K-bounded family of subsets of U is a family $B = (B_v)_{v \in M(\overline{K})}$ such that

(a) for each $v \in M(\overline{K})$, $B_v \subset U(\overline{K})$ is a bounded subset with respect to U and v;

(b) for any $f \in \mathcal{O}(U)$ and for any $v_0 \in M_K$,

$$C_{v_0,B}(f) := \sup_{\substack{v \in M(\overline{K}) \\ v|v_0}} \sup_{x \in B_v}\{|f(x)|_v\} < \infty$$

and $C_{v_0,B}(f) \leqslant 1$ for all but finitely many $v_0 \in M_K$.

Remark 4.12. Let $V \subset U \subset X$ are open affine subsets of X. Let $v \in M$. Then a bounded subset with respect to V and v is a bounded subset with respect to U and v. Also, an M_K-bounded family of subsets of V is an M_K-bounded family of subsets of U.

Remark 4.13. Let $U \subset X$ be an open subset. Finite unions of M_K-bounded family of subsets of U are also M_K-bounded family of subsets of U.

Lemma 4.14. *Let $K \subset L \subset \overline{K}$ be an intermediate field such that $[L : K] < \infty$. Let X be an algebraic scheme over L. Let $U \subset X$ be an affine open subset and $B = (B_v)_{v \in M(\overline{K})}$ be an M_K-bounded family of subsets of U.*

Let $\{U_i\}_{i=1}^r$ be an open cover of U where U_i are principal open subsets of U. Then for each i, there is an M_K-bounded family of subsets $B^i = (B^i_v)_{v \in M(\overline{K})}$ of U_i such that

$$\bigcup_{i=1}^r B^i_v = B_v \quad \textit{for every } v \in M(\overline{K}).$$

Proof. Let $U = \operatorname{Spec} A$ and $U_i = \operatorname{Spec} A_{f_i}$ where $f_i \in A$. Since $\bigcup_{i=1}^r U_i = U$, there are $a_i, \ldots, a_r \in A$ such that

$$a_1 f_1 + \cdots + a_r f_r = 1.$$

Take any $v \in M(\overline{K})$ and $x \in B_v$. Let $v_0 = v|_K \in M_K$. Then

$$1 = |a_1 f_1 + \cdots + a_r f_r|_v \leqslant r^{\epsilon(v_0)} \max_{1\leqslant i\leqslant r}\{C_{v_0,B}(a_i)\} \max_{1\leqslant i\leqslant r}\{|f_i(x)|_v\}.$$

Thus we have

$$\frac{1}{\max_{1\leqslant i\leqslant r}\{|f_i(x)|_v\}} \leqslant r^{\epsilon(v_0)} \max_{1\leqslant i\leqslant r}\{C_{v_0,B}(a_i)\}.$$

Now, let $B_v^i = \{x \in B_v \mid |f_i(x)|_v = \max_{1\leqslant j\leqslant r}\{|f_j(x)|_v\}\}$. Obviously, we have $\bigcup_{i=1}^r B_v^i = B_v$. For every x, one of the $f_i(x)$ is not zero, and hence $B_v^i \subset U_i(\overline{K})$. Let $\varphi \in \mathcal{O}(U_i)$. Then we can write $\varphi = \alpha/f_i^n$ for some $\alpha \in A$ and $n \geqslant 0$. Take any $v \in M(\overline{K})$ and let $v_0 = v|_K \in M_K$. Then for $x \in B_v^i$ we have

$$\begin{aligned} |\varphi(x)|_v = \frac{|\alpha(x)|_v}{|f_i(x)|_v^n} &= \frac{|\alpha(x)|_v}{\max_{1\leqslant j\leqslant r}\{|f_j(x)|_v\}^n} \\ &\leqslant r^{\epsilon(v_0)n} \max_{1\leqslant i\leqslant r}\{C_{v_0,B}(a_i)\}^n C_{v_0,B}(\alpha). \end{aligned}$$

Thus B_v^i is bounded with respect to U_i and v. Moreover

$$\sup_{v|v_0} \sup_{x\in B_v^i}\{|\varphi(x)|_v\} \leqslant r^{\epsilon(v_0)n} \max_{1\leqslant i\leqslant r}\{C_{v_0,B}(a_i)\}^n C_{v_0,B}(\alpha)$$

and the right hand side is $\leqslant 1$ for all but finitely many v_0. This proves $(B_v^i)_{v\in M(\overline{K})}$ is an M_K-bounded family of subsets of U_i.

□

Proposition 4.15. *Let $K \subset L \subset \overline{K}$ be an intermediate field such that $[L : K] < \infty$. Let X be a projective scheme over L. Let $\{U_i\}_{i=1}^r$ be an open affine cover of X. Then there are M_K-bounded family of subsets $(B_v^i)_{v\in M(\overline{K})}$ of U_i such that*

$$\bigcup_{i=1}^r B_v^i = X(\overline{K}) \quad \textit{for every } v \in M(\overline{K}).$$

Proof. By Lemma 4.14, it is enough to show the proposition for one specific open affine cover $\{U_i\}_{i=1}^r$ of X.

Fix an embedding $\iota\colon X \longrightarrow \mathbb{P}_L^N = \operatorname{Proj} L[x_0, \ldots, x_N]$. Set $U_i = \iota^{-1}(D_+(x_i))$ and $\varphi_{ij} = \iota^*(x_j/x_i) \in \mathcal{O}(U_i)$. Let

$$B_v^i = \{x \in U_i(\overline{K}) \mid |\varphi_{ij}(x)|_v \leqslant 1 \text{ for all } j = 0, \ldots, N\}.$$

Since $\varphi_{ij}, j = 0, \ldots, N$ generate $\mathcal{O}(U_i)$ as an L-algebra, B_v^i is bounded with respect to U_i and v. Take any $f \in \mathcal{O}(U_i)$. Then there is a polynomial P in $N+1$-variables with coefficient in L such that $f = P(\varphi_{i0}, \ldots, \varphi_{iN})$. Then for any $v_0 \in M_K$, we have

$$\begin{aligned}\sup_{\substack{v|v_0 \\ v \in M}} \sup_{x \in B_v^i} \{|f(x)|_v\} &\leqslant \sup_{v|v_0} \{T^{\epsilon(v_0)} \max_{a \text{ coeff. of } P} \{|a|_v\}\} \\ &\leqslant T^{\epsilon(v_0)} \sup_{\substack{w \in M(L) \\ w|v_0}} \{\max\{|a|_w \mid a \text{ coeff. of } P\}\}\end{aligned}$$

where T is the number of terms in P. The right hand side is $\leqslant 1$ for all but finitely many v_0. Therefore $(B_v^i)_{v \in M}$ is an M_K-bounded family of subsets of U_i.

Finally we prove $\bigcup_{i=0}^N B_v^i = X(\overline{K})$. Fix $v \in M$. Let $x \in X(\overline{K})$. Write $\iota(x) = (a_0 : \cdots : a_N)$. For i such that $|a_i|_v = \max_{0 \leqslant j \leqslant N}\{|a_j|_v\}$, we have

$$|\varphi_{ij}(x)|_v = \frac{|a_j|_v}{|a_i|_v} \leqslant 1 \quad \text{for } j = 0, \ldots, N.$$

Thus $x \in B_v^i$ and this proves $\bigcup_{i=0}^N B_v^i = X(\overline{K})$.

□

4.3 Local height associated to presentations of closed subschemes

In this section we fix an infinite field K equipped with a proper set of absolute values M_K. We fix an algebraic closure $\overline{K}$ of K and let

$$M = M(\overline{K}) = \left\{ |\ |_v \;\middle|\; \begin{array}{c} |\ |_v \text{ is an absolute value on } \overline{K} \text{ such that} \\ \text{the restriction } |\ |_v|_K \text{ on } K \text{ is an element of } M_K \end{array} \right\}.$$

Let $K \subset F \subset L \subset \overline{K}$ be intermediate fields such that $[L : K] < \infty$. (The field F is going to be the field over which our ambient scheme X is defined and L is going to be the one over which the closed subschemes, to which we will associate height functions, are defined.)

Remark 4.16. For any F-scheme X, there is a canonical injection $X(L) \longrightarrow X(\overline{K})$ induced by the inclusion $L \subset \overline{K}$. By this manner, we identify $X(L)$ as a subset of $X(\overline{K})$.

We fix a quasi-projective scheme X over F in the rest of this section.

Definition 4.17 (Local height associated with presentations). Let $Y \subset X_L$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X_L) = \varnothing$. Let $\mathcal{Y} = (Y; \mathcal{D}_1, \ldots, \mathcal{D}_r)$ be a presentation of Y where

$$\mathcal{D}_i = (s_{D_i}; \mathcal{L}^{(i)}, s_0^{(i)}, \ldots s_{n_i}^{(i)}; \mathcal{M}^{(i)}, t_0^{(i)}, \ldots, t_{m_i}^{(i)}; \psi^{(i)}).$$

We define the map, which is called local height function associated with presentation $\mathcal{Y}$,

$$\lambda_{\mathcal{Y}} \colon (X_L \backslash Y)(\overline{K}) \times M(\overline{K}) \longrightarrow \mathbb{R}$$

as follows. For $(x, v) \in (X_L \backslash Y)(\overline{K}) \times M(\overline{K})$, we set

$$\lambda_{\mathcal{Y}}(x, v) := \min\{\lambda_{\mathcal{D}_1}(x, v), \ldots, \lambda_{\mathcal{D}_r}(x, v)\}$$

where

$$\lambda_{\mathcal{D}_i}(x, v) := \log\left(\max_{0 \leqslant j \leqslant n_i}\left\{\min_{0 \leqslant k \leqslant m_i}\left\{\left|\frac{s_j^{(i)}}{s_{D_i} t_k^{(i)}}(x)\right|_v\right\}\right\}\right).$$

Here $s_j^{(i)}/s_{D_i} t_k^{(i)}$ is an element of $\mathcal{O}_X(X \backslash (D_i \cup \mathrm{div}(t_k^{(i)})))$ defined in Construction 4.10. If $x \notin (X \backslash (D_i \cup \mathrm{div}(t_k^{(i)})))(\overline{K})$, we set $|s_j^{(i)}/s_{D_i} t_k^{(i)}(x)|_v = \infty$. We use the conventions $a < \infty$ for all $a \in \mathbb{R}$ and $\log(\infty) = \infty$.

Remark 4.18. Sometime it is convenient to allow $x \in Y(\overline{K})$. We define $\lambda_{\mathcal{Y}}(x, v) = \infty$ if $x \in Y(\overline{K})$.

Lemma 4.19. *Let D, E be two effective Cartier divisors on X_L and let $\mathcal{D}, \mathcal{E}$ be presentations of D, E. Then*

$$\lambda_{\mathcal{D}+\mathcal{E}} = \lambda_{\mathcal{D}} + \lambda_{\mathcal{E}} \colon (X_L \backslash (D \cup E)) \times M(\overline{K}) \longrightarrow \mathbb{R}.$$

Proof. Let

$$\begin{aligned}\mathcal{D} &= (s_D; \mathcal{L}, s_0, \ldots, s_n; \mathcal{M}, t_0, \ldots, t_m; \psi),\\ \mathcal{E} &= (s_E; \mathcal{L}', s_0', \ldots, s_{n'}'; \mathcal{M}', t_0', \ldots, t_{m'}'; \psi').\end{aligned}$$

Recall the sum $\mathcal{D} + \mathcal{E}$ is defined as

$$\mathcal{D} + \mathcal{E} = \Big(s_{D+E}; \mathcal{L} \otimes \mathcal{L}', \{s_i \otimes s_j'\}_{\substack{0 \leqslant i \leqslant n \\ 0 \leqslant j \leqslant n'}}; \mathcal{M} \otimes \mathcal{M}', \{t_k \otimes t_l'\}_{\substack{0 \leqslant k \leqslant m \\ 0 \leqslant l \leqslant m'}}; \psi \otimes \psi'\Big).$$

Note that

$$\frac{s_i \otimes s'_j}{s_{D+E} t_k \otimes t'_l} = \frac{s_i}{s_D t_k} \frac{s'_j}{s_E t'_l}$$

as elements of $\mathcal{O}_{X_L}(X_L \backslash (D \cup E \cup \operatorname{div}(t_k) \cup \operatorname{div}(t'_l)))$. Then for $(x, v) \in (X_L \backslash (D \cup E)) \times M(\overline{K})$,

$$\begin{aligned} \lambda_{\mathcal{D}+\mathcal{E}}(x, v) &= \log \left(\max_{i,j} \min_{k,l} \left\{ \left| \frac{s_i}{s_D t_k}(x) \frac{s'_j}{s_E t'_l}(x) \right|_v \right\} \right) \\ &= \log \left(\max_i \min_k \left\{ \left| \frac{s_i}{s_D t_k}(x) \right|_v \right\} \max_j \min_l \left\{ \left| \frac{s'_j}{s_E t'_l}(x) \right|_v \right\} \right) \\ &= \lambda_{\mathcal{D}}(x, v) + \lambda_{\mathcal{E}}(x, v). \end{aligned}$$

□

Lemma 4.20. *Let D be an effective Cartier divisor on X_L and let $\mathcal{D}$ be a presentation of D. Let X' be a quasi-projective scheme over L and $f \colon X' \longrightarrow X_L$ a morphism over L. Suppose $f^*\mathcal{D}$ is well-defined. Then*

$$\lambda_{\mathcal{D}} \circ (f \times \mathrm{id}) = \lambda_{f^*\mathcal{D}} \colon (X' \backslash f^{-1}(D))(\overline{K}) \times M(\overline{K}) \longrightarrow \mathbb{R}.$$

Proof. Let

$$\mathcal{D} = (s_D; \mathcal{L}, s_0, \ldots, s_n; \mathcal{M}, t_0, \ldots, t_m; \psi).$$

For $(x, v) \in (X' \backslash f^{-1}(D))(\overline{K}) \times M(\overline{K})$, we have

$$\begin{aligned} \lambda_{f^*\mathcal{D}}(x, v) &= \log \max_i \min_j \left\{ \left| \frac{f^* s_i}{s_{f^*D} f^* t_j}(x) \right|_v \right\} \\ &= \log \max_i \min_j \left\{ \left| \frac{s_i}{s_D t_j}(f(x)) \right|_v \right\} \\ &= \lambda_{\mathcal{D}}(f(x), v). \end{aligned}$$

□

Proposition 4.21 (Basic properties of local height functions associated with presentations)**.** *Let $Y, W \subset X_L$ be closed subschemes such that $Y \cap \operatorname{Ass}(X) = W \cap \operatorname{Ass}(X) = \varnothing$. Let $\mathcal{Y}, \mathcal{W}$ be presentations of Y, W. Then the following hold:*

(1) $\lambda_{\mathcal{Y}\cap\mathcal{W}} = \min\{\lambda_{\mathcal{Y}}, \lambda_{\mathcal{W}}\}\colon (X_L\backslash(Y \cap W))(\overline{K}) \times M(\overline{K}) \longrightarrow \mathbb{R}$;

(2) $\lambda_{\mathcal{Y}+\mathcal{W}} = \lambda_{\mathcal{Y}} + \lambda_{\mathcal{W}}\colon (X_L\backslash(Y \cup W))(\overline{K}) \times M(\overline{K}) \longrightarrow \mathbb{R}$;

(3) *Let $L \subset L' \subset \overline{K}$ be an intermediate field such that $[L' : L] < \infty$. Then*

$$\begin{array}{rl} \lambda_{\mathcal{Y}} = \lambda_{\mathcal{Y}_{L'}}\colon & (X_L\backslash Y)(\overline{K}) \times M(\overline{K}) \longrightarrow \mathbb{R}. \\ & \quad\| \\ & (X_{L'}\backslash Y_{L'})(\overline{K}) \times M(\overline{K}) \end{array}$$

(4) *Let X' be a quasi-projective scheme over L and $f\colon X' \longrightarrow X$ a morphism over L. Suppose $f^*\mathcal{Y}$ is well-defined. Then*

$$\lambda_{\mathcal{Y}} \circ (f \times \mathrm{id}) = \lambda_{f^*\mathcal{Y}}\colon (X'\backslash f^{-1}(Y))(\overline{K}) \times M(\overline{K}) \longrightarrow \mathbb{R}.$$

Proof. Let $\mathcal{Y} = (Y; \mathcal{D}_1, \ldots, \mathcal{D}_r)$ and $\mathcal{W} = (W; \mathcal{E}_1, \ldots, \mathcal{E}_s)$.

(1) Since $\mathcal{Y} \cap \mathcal{W} = (Y \cap W; \mathcal{D}_1, \ldots, \mathcal{D}_r, \mathcal{E}_1, \ldots, \mathcal{E}_s)$, we have

$$\lambda_{\mathcal{Y}\cap\mathcal{W}} = \min_{i,j}\{\lambda_{D_i}, \lambda_{\mathcal{E}_j}\} = \min\{\lambda_{\mathcal{Y}}, \lambda_{\mathcal{W}}\}.$$

(2) Since $\mathcal{Y} + \mathcal{W} = (Y + W; \{\mathcal{D}_i + \mathcal{E}_j\}_{i,j})$, by Lemma 4.19 we have

$$\lambda_{\mathcal{Y}+\mathcal{W}} = \min_{i,j}\{\lambda_{\mathcal{D}_i+\mathcal{E}_j}\} = \min_{i,j}\{\lambda_{\mathcal{D}_i} + \lambda_{\mathcal{E}_j}\} = \lambda_{\mathcal{Y}} + \lambda_{\mathcal{W}}.$$

(3) This is trivial from the definition.

(4) Since $f^*\mathcal{Y} = (f^{-1}(Y); f^*\mathcal{D}_1, \ldots, f^*\mathcal{D}_r)$, by Lemma 4.20 we have

$$\begin{aligned} \lambda_{f^*\mathcal{Y}} &= \min\{\lambda_{f^*\mathcal{D}_1}, \ldots, \lambda_{f^*\mathcal{D}_r}\} \\ &= \min\{\lambda_{\mathcal{D}_1} \circ (f \times \mathrm{id}), \ldots, \lambda_{\mathcal{D}_1} \circ (f \times \mathrm{id})\} = \lambda_{\mathcal{Y}} \circ (f \times \mathrm{id}). \end{aligned}$$

□

Lemma 4.22. *Let $Y \subset X_L$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Let $\mathcal{Y}$ be a presentation of Y. Let $L \subset L' \subset \overline{K}$ be an intermediate field. Let $x \in X(L')$ be any point. For $v, w \in M(\overline{K})$, if $v|_{L'} = w|_{L'}$, then*

$$\lambda_{\mathcal{Y}}(x, v) = \lambda_{\mathcal{Y}}(x, w).$$

Proof. In the notation of Definition 4.17,

$$\frac{s_j^{(i)}}{s_{D_i} t_k^{(i)}}(x)$$

is contained in L'. Thus

$$\left|\frac{s_j^{(i)}}{s_{D_i} t_k^{(i)}}(x)\right|_v = \left|\frac{s_j^{(i)}}{s_{D_i} t_k^{(i)}}(x)\right|_w$$

and we get $\lambda_{\mathcal{Y}}(x, v) = \lambda_{\mathcal{Y}}(x, w)$. □

Definition 4.23 (M_K-constant)**.** A map $\gamma\colon M(\overline{K}) \longrightarrow \mathbb{R}_{\geqslant 0}$ is called M_K-constant if γ factors through M_K, i.e.

$$\begin{tikzcd} M(\overline{K}) \arrow[r, "\gamma"] \arrow[d, "\text{restriction}"'] & \mathbb{R}_{\geqslant 0} \\ M_K \arrow[ur, "\exists\overline{\gamma}"'] & \end{tikzcd}$$

and $\overline{\gamma}(v) = 0$ for all but finitely many $v \in M_K$. We denote $\overline{\gamma}$ also by γ for simplicity.

For any intermediate field $K \subset L \subset \overline{K}$ with $[L : K] < \infty$, we also write $\gamma(v) = \overline{\gamma}(v|_K)$ for $v \in M(L)$. Note that $\gamma(v) = 0$ for all but finitely many $v \in M(L)$.

We have concrete expressions of local heights when they are restricted on M_K-bounded families of subsets.

Lemma 4.24. *Let $D \subset X_L$ be an effective Cartier divisor and $\mathcal{D}$ be a presentation of D. Let $U \subset X_L$ be a non-empty open affine subset such that*

$$\begin{aligned} &D|_U = \operatorname{div}(f) \quad \text{for a non-zero divisor } f \in \mathcal{O}(U); \\ &\mathcal{L}|_U \simeq \mathcal{O}_U \simeq \mathcal{M}|_U. \end{aligned}$$

Let $B = (B_v)_{v \in M(\overline{K})}$ be an M_K-bounded family of subsets of U. Then there exists an M_K-constant γ such that

$$\log \frac{1}{|f(x)|_v} - \gamma(v) \leqslant \lambda_{\mathcal{D}}(x, v) \leqslant \log \frac{1}{|f(x)|_v} + \gamma(v)$$

for all $v \in M(\overline{K})$ and $x \in B_v \backslash D(\overline{K})$.

Proof. Let $\mathcal{D} = (s_D; \mathcal{L}, s_0, \ldots, s_n; \mathcal{M}, t_0. \ldots, t_m; \psi)$. Fix isomorphisms

$$\begin{aligned} \alpha &\colon \mathcal{L}|_U \xrightarrow{\sim} \mathcal{O}_U; \\ \beta &\colon \mathcal{M}|_U \xrightarrow{\sim} \mathcal{O}_U \end{aligned}$$

and set

$$\begin{aligned} \alpha(s_i|_U) &= g_i; \\ \beta(t_j|_U) &= h_j. \end{aligned}$$

Note that g_i, h_j are non-zero divisors of $\mathcal{O}(U)$. We also have the isomorphism

$$\alpha \otimes (\beta^\vee)^{-1} \circ \psi^{-1} \colon \mathcal{O}_X(D)|_U \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{M}^{-1}|_U \xrightarrow{\sim} \mathcal{O}_U.$$

Then $\alpha \otimes \beta^\vee \circ \psi^{-1}(s_D|_U) = uf$ for some $u \in \mathcal{O}(U)^\times$. Then on $V = U \backslash (D \cup \operatorname{div}(t_j))$, we have

$$\begin{array}{ccc} \mathcal{L} \otimes \mathcal{M}^{-1} \otimes \mathcal{O}_X(D)^{-1}|_V \xrightarrow{\psi\otimes \mathrm{id}} \mathcal{O}_X(D) \otimes \mathcal{O}_X(D)^{-1}|_V & \xrightarrow{\sim} & \mathcal{O}_V \\ s_i|_V \otimes t_j^{-1}|_V \otimes s_D^{-1}|_V & \longmapsto & \left.\dfrac{s_i}{s_D t_j}\right|_V = \dfrac{g_i}{ufh_j}. \end{array}$$

Since $s_0, \ldots, s_n$ generate $\mathcal{L}$, there are $a_0, \ldots, a_n \in \mathcal{O}(U)$ such that

$$a_0 g_0 + \cdots + a_n g_n = 1. \tag{4.1}$$

Therefore for any $v \in M(\overline{K})$ and $x \in B_v$, we have

$$\begin{aligned} 1 &= |a_0(x)g_0(x) + \cdots + a_n(x)g_n(x)|_v \\ &\leqslant (n+1)^{\epsilon(v)} \max_{0\leqslant i\leqslant n}\{|a_i(x)|_v\} \max_{0\leqslant i\leqslant n}\{|g_i(x)|_v\} \\ &\leqslant (n+1)^{\epsilon(v)} \max_{0\leqslant i\leqslant n}\{\sup_{v'|v_0}\sup_{y\in B_v}\{|a_i(y)|_{v'}\}\} \max_{0\leqslant i\leqslant n}\{|g_i(x)|_v\} \quad \text{where } v_0 = v|_K \\ &= (n+1)^{\epsilon(v_0)} \max_{0\leqslant i\leqslant n}\{C_{v_0,B}(a_i)\} \max_{0\leqslant i\leqslant n}\{|g_i(x)|_v\}. \end{aligned}$$

By (4.1), $\max_{0\leqslant i\leqslant n}\{C_{v_0,B}(a_i)\} \neq 0$ and we get

$$\max_{0\leqslant i\leqslant n}\{|g_i(x)|_v\} \geqslant \frac{1}{(n+1)^{\epsilon(v_0)} \max_{0\leqslant i\leqslant n}\{C_{v_0,B}(a_i)\}}. \tag{4.2}$$

Similarly, we take $b_j \in \mathcal{O}(U)$ so that $b_0h_0 + \cdots + b_mh_m = 1$ and get

$$\max_{0\leqslant j\leqslant m}\{|h_j(x)|_v\} \geqslant \frac{1}{(m+1)^{\epsilon(v_0)} \max_{0\leqslant j\leqslant m}\{C_{v_0,B}(b_j)\}}. \tag{4.3}$$

For any $v \in M(\overline{K})$ and $x \in B_v \backslash D(\overline{K})$, let us set $v_0 = v|_K$, then by (4.2) and (4.3),

$$\max_{0\leqslant i\leqslant n}\min_{0\leqslant j\leqslant m}\left|\frac{s_i}{s_D t_j}(x)\right|_v = \max_{0\leqslant i\leqslant n}\min_{0\leqslant j\leqslant m}\left|\frac{g_i(x)}{u(x)f(x)h_j(x)}\right|_v$$

$$\begin{cases} \leqslant C_{v_0,B}(u^{-1})(m+1)^{\epsilon(v_0)} \max\limits_{0\leqslant j\leqslant m}\{C_{v_0,B}(b_j)\} \max\limits_{0\leqslant i\leqslant n}\{C_{v_0,B}(g_i)\} \dfrac{1}{|f(x)|_v} \\ \geqslant \dfrac{1}{C_{v_0,B}(u) \max\limits_{0\leqslant j\leqslant m}\{C_{v_0,B}(h_j)\}(n+1)^{\epsilon(v_0)} \max\limits_{0\leqslant i\leqslant n}\{C_{v_0,B}(a_i)\}} \dfrac{1}{|f(x)|_v}. \end{cases}$$

Since the big constant coefficients are $\leqslant 1$ and $\geqslant 1$ for all but finitely many $v_0 \in M_K$ respectively, we are done.

$\square$

On projective schemes, local height functions associated with presentations essentially depend only on the closed subschemes.

Proposition 4.25 (Independence on presentations)**.** *Suppose X is projective over F. Let $Y \subset X_L$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Let $\mathcal{Y}$ and $\mathcal{Y}'$ be two presentations of Y. Then there exists an M_K-constant γ such that*

$$\lambda_{\mathcal{Y}}(x,v) - \gamma(v) \leqslant \lambda_{\mathcal{Y}'}(x,v) \leqslant \lambda_{\mathcal{Y}}(x,v) + \gamma(v)$$

for all $(x,v) \in (X_L \backslash Y)(\overline{K}) \times M(\overline{K})$.

Proof. Let $\mathcal{Y} = (Y; \mathcal{D}_1, \ldots, \mathcal{D}_r)$ and $\mathcal{Y}' = (Y; \mathcal{E}_1, \ldots, \mathcal{E}_s)$. Then $\mathcal{Y}'' = (Y; \mathcal{D}_1, \ldots, \mathcal{D}_r, \mathcal{E}_1, \ldots, \mathcal{E}_s)$ is also a presentation of Y. By replacing $\mathcal{Y}'$ with $\mathcal{Y}''$, we may assume $\mathcal{Y}' = (Y; \mathcal{D}_1, \ldots, \mathcal{D}_r, \mathcal{E}_1, \ldots, \mathcal{E}_s)$. By arguing inductively on s, we may assume $\mathcal{Y}' = (Y; \mathcal{D}_1, \ldots, \mathcal{D}_r, \mathcal{E})$.

By definition, we have $\lambda_{\mathcal{Y}} \geqslant \lambda_{\mathcal{Y}'}$. To end the proof, it is enough to show that $\lambda_{\mathcal{Y}'} \geqslant \lambda_{\mathcal{Y}} - \gamma$ for some M_K-constant γ.

Claim 4.26. *Let $U \subset X_L$ be an open affine subset such that all invertible $\mathcal{O}_X$-modules in $\mathcal{D}_1, \ldots, \mathcal{D}_r, \mathcal{E}$ are isomorphic to $\mathcal{O}_U$. Let $B = (B_v)_{v \in M(\overline{K})}$ be an M_K-bounded family of subsets of U. Then there exists an M_K-constant $\gamma_{U,B}$ such that*

$$\lambda_{\mathcal{Y}'}(x,v) \geqslant \lambda_{\mathcal{Y}}(x,v) - \gamma_{U,B}(v)$$

for all $v \in M(\overline{K})$ and $x \in B_v \backslash Y(\overline{K})$.

Proof. Let $D_1, \ldots, D_r, E$ be the effective Cartier divisors presented by $\mathcal{D}_1$, $\ldots, \mathcal{D}_r, \mathcal{E}$. Let $f_1, \ldots, f_r, g \in \mathcal{O}(U)$ be non-zero divisors such that

$$D_1|_U = \mathrm{div}(f_1), \ldots, D_r|_U = \mathrm{div}(f_r), E|_U = \mathrm{div}(g).$$

Since $\mathcal{D}_1, \ldots \mathcal{D}_r$ already form a presentation of Y, we can write

$$g = \sum_{i=1}^{r} a_i f_i$$

for some $a_i \in \mathcal{O}(U)$. Then for any $v \in M(\overline{K})$ and $x \in B_v$, set $v_0 = v|_K$ and get

$$|g(x)|_v \leqslant r^{\epsilon(v_0)} \max_{1\leqslant i\leqslant r}\{C_{v_0,B}(a_i)\} \max_{1\leqslant i\leqslant r}\{|f_i(x)|_v\}.$$

Therefore, if $x \in B_v \backslash Y(\overline{K})$, we have

$$\begin{aligned}
\lambda_{\mathcal{Y}'}(x,v) &= \min\{\lambda_{\mathcal{D}_1}(x,v),\ldots,\lambda_{\mathcal{D}_r}(x,v),\lambda_{\mathcal{E}}(x,v)\} \\
&\geqslant \min\left\{\log\frac{1}{|f_1(x)|_v},\ldots,\log\frac{1}{|f_r(x)|_v},\log\frac{1}{|g(x)|_v}\right\} - \gamma(v) \\
&\qquad\qquad \text{for some } M_K\text{-constant } \gamma \\
&\qquad\qquad (\text{ Lemma 4.24}) \\
&\geqslant \min\left\{\log\frac{1}{|f_1(x)|_v},\ldots,\log\frac{1}{|f_r(x)|_v},\log\frac{1}{\max_{1\leqslant i\leqslant r}\{|f_i(x)|_v\}}\right\} \\
&\quad - \log\Big(r^{\epsilon(v_0)}\max_{1\leqslant i\leqslant r}\{C_{v_0,B}(a_i)\}\Big) - \gamma(v) \\
&= \min_{1\leqslant i\leqslant r}\left\{\log\frac{1}{|f_i(x)|_v}\right\} - \log\Big(r^{\epsilon(v_0)}\max_{1\leqslant i\leqslant r}\{C_{v_0,B}(a_i)\}\Big) - \gamma(v) \\
&\geqslant \lambda_{\mathcal{Y}}(x,v) - \gamma'(v) - \log\Big(r^{\epsilon(v_0)}\max_{1\leqslant i\leqslant r}\{C_{v_0,B}(a_i)\}\Big) - \gamma(v) \\
&\qquad\qquad \text{for some } M_K\text{-constant } \gamma' \\
&\qquad\qquad (\text{ Lemma 4.24})
\end{aligned}$$

and we are done.

■

Now, take a finite open affine cover $\{U_i\}_{i=1}^n$ of X consisting of open affines as in the claim. By Proposition 4.15, there are M_K-bounded family of subsets $B^i = (B^i_v)_{v\in M(\overline{K})}$ of U_i such that $\bigcup_{i=1}^n B^i_v = X(\overline{K})$ for each $v \in M(\overline{K})$. Then for any $v \in M(\overline{K})$ and $x \in (X_L\backslash Y)(\overline{K})$, we have

$$\lambda_{\mathcal{Y}'}(x,v) \geqslant \lambda_{\mathcal{Y}}(x,v) - \max_{1\leqslant i\leqslant n}\{\gamma_{U_i,B^i}(v)\}.$$

□

4.4 Local height associated to closed subschemes

In this section we fix an infinite field K equipped with a proper set of absolute values M_K. We fix an algebraic closure $\overline{K}$ of K and let

$$M = M(\overline{K}) = \left\{ |\ |_v \,\middle|\, \begin{array}{c} |\ |_v \text{ is an absolute value on } \overline{K} \text{ such that} \\ \text{the restriction } |\ |_v|_K \text{ on } K \text{ is an element of } M_K \end{array} \right\}.$$

We will associate local height functions (up to M_K-bounded function) to closed subschemes of projective schemes over $\overline{K}$.

Definition 4.27. Let U be a quasi-projective scheme over $\overline{K}$. Two maps

$$\lambda_1, \lambda_2 \colon U(\overline{K}) \times M(\overline{K}) \longrightarrow \mathbb{R}$$

are said to be equal up to M_K-bounded function if there is an M_K-constant γ such that

$$\lambda_1(x, v) - \gamma(v) \leqslant \lambda_2(x, v) \leqslant \lambda_1(x, v) + \gamma(v)$$

for all $(x, v) \in U(\overline{K}) \times M(\overline{K})$. Note that this defines an equivalence relation. Any function in the equivalence class of the constant map 0 is called M_K-bounded function. When λ_1 and λ_2 are equal up to M_K-bounded function, we write $\lambda_1 = \lambda_2 + O_{M_K}(1)$.

When there is an M_K-constant γ such that

$$\lambda_1(x, v) \leqslant \lambda_2(x, v) + \gamma(v)$$

for all $(x, v) \in U(\overline{K}) \times M(\overline{K})$, we write $\lambda_1 \leqslant \lambda_2 + O_{M_K}(1)$.

Remark 4.28. When the functions are allowed to take value ∞, we use the same notation under the convention:

- $a \leqslant \infty$ for all $a \in \mathbb{R}$;
- $\infty \pm a = \infty$ for all $a \in \mathbb{R}$;
- $\infty + \infty = \infty$
- $\infty - \infty = 0$.

In particular,

(1) if $\lambda_1 \leqslant \lambda_2 + O_{M_K}(1)$, then $\lambda_1(x, v) = \infty$ implies $\lambda_2(x, v) = \infty$.

(2) if $\lambda_1 = \lambda_2 + O_{M_K}(1)$, then $\lambda_1(x, v) = \infty$ if and only if $\lambda_2(x, v) = \infty$.

Construction 4.29. Let X be a projective scheme over $\overline{K}$. Let $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Take an intermediate field $K \subset L \subset \overline{K}$ such that

- $[L : K] < \infty$;
- there is a projective scheme X_L over L and a closed subscheme $Y_L \subset X_L$;
- there is an isomorphism $(X_L)_{\overline{K}} \simeq X$ of $\overline{K}$-schemes by which $(Y_L)_{\overline{K}}$ is isomorphic to Y:

$$\begin{tikzcd} (X_L)_{\overline{K}} \arrow[r, "\sim"] & X \\ (Y_L)_{\overline{K}} \arrow[u, hook] \arrow[r, "\sim"] & Y \arrow[u, hook] \end{tikzcd}$$

We identify $(X_L)_{\overline{K}}, (Y_L)_{\overline{K}}$ with X, Y via this isomorphisms. Note that since $X \longrightarrow X_L$ is flat, we have $Y_L \cap \mathrm{Ass}(X_L)$. Thus we can take a presentation $\mathcal{Y}$ of Y_L and get a map

$$\lambda_{\mathcal{Y}} \colon (X \backslash Y)(\overline{K}) \times M(\overline{K}) \longrightarrow \mathbb{R}.$$

Let $L', X_{L'}, Y_{L'}$ be another such a field and schemes and take a presentation $\mathcal{Y}'$ of $Y_{L'}$. Then there is a subfield $L'' \subset \overline{K}$ that contains L, L', is finite over K, and the isomorphism $(X_L)_{\overline{K}} \xrightarrow{\sim} X \xrightarrow{\sim} (X_{L'})_{\overline{K}}$ is defined over L''. Then we get isomorphisms of L''-schemes

$$\begin{tikzcd} (X_L)_{L''} \arrow[r, "\sim"] & (X_{L'})_{L''} \\ (Y_L)_{L''} \arrow[u, hook] \arrow[r, "\sim"] & (Y_{L'})_{L''} \arrow[u, hook] \end{tikzcd}$$

and identify them by these isomorphisms. Then $\mathcal{Y}_{L''}$ and $\mathcal{Y}'_{L''}$ are presentations of the same closed subscheme and therefore by Proposition 4.21(3) and Proposition 4.25, there exists an M_K-constant γ such that

$$\lambda_{\mathcal{Y}}(x, v) - \gamma(v) \leqslant \lambda_{\mathcal{Y}'}(x, v) \leqslant \lambda_{\mathcal{Y}}(x, v) + \gamma(v)$$

for all $(x, v) \in (X \backslash Y)(\overline{K}) \times M(\overline{K})$.

Therefore $\lambda_{\mathcal{Y}}$ and $\lambda_{\mathcal{Y}'}$ are equal up to M_K-bounded function.

Definition 4.30 (Local height associated with closed subschemes). In the notation of Construction 4.29, any map $(X \backslash Y)(\overline{K}) \times M(\overline{K}) \longrightarrow \mathbb{R}$ which is equal to $\lambda_{\mathcal{Y}}$ up to M_K-bounded function is called a height function associated with Y and denoted by λ_Y. This is well-defined by Construction 4.29.

Remark 4.31. Note that λ_Y is determined up to M_K-bounded function by Y.

Proposition 4.32 (Good choice of local heights)**.** *Let $K \subset F \subset \overline{K}$ be an intermediate field such that $[F : K] < \infty$. Let X be a projective scheme over F. Let $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Then we can take a local height function $\lambda_{Y_{\overline{K}}}$ associated with $Y_{\overline{K}}$ so that the following holds. For any intermediate field $F \subset L \subset \overline{K}$ with $[L : F] < \infty$, $\lambda_{Y_{\overline{K}}}$ induces a map $(X_L \backslash Y_L)(L) \times M(L) \longrightarrow \mathbb{R}$:*

$$\begin{tikzcd}
(X_L \backslash Y_L)(L) \times M(\overline{K}) \arrow[r, hook] \arrow[d, "\mathrm{id} \times (\ )|_L"'] & (X_{\overline{K}} \backslash Y_{\overline{K}})(\overline{K}) \times M(\overline{K}) \arrow[r, "\lambda_{Y_{\overline{K}}}"] & \mathbb{R} \\
(X_L \backslash Y_L)(L) \times M(L) \arrow[urr, bend right=10, "\exists"'] & &
\end{tikzcd}.$$

Namely, $\lambda_{Y_{\overline{K}}} = \lambda_{\mathcal{Y}}$ works for any presentation $\mathcal{Y}$ of Y.

Proof. This follows from Lemma 4.22. □

Let us prove basic properties of local height functions. Let us first recall operations of closed subschemes. Let X be an algebraic scheme and $Y, W \subset X$ be closed subschemes and $\mathcal{I}_Y, \mathcal{I}_W$ be ideals of them. We define:

(1) $Y \cap W \subset X$ is the closed subscheme defined by $\mathcal{I}_Y + \mathcal{I}_W$;

(2) $Y + W \subset X$ is the closed subscheme defined by $\mathcal{I}_Y \mathcal{I}_W$;

(3) $Y \cup W \subset X$ is the closed subscheme defined by $\mathcal{I}_Y \cap \mathcal{I}_W$.

Note that
$$\mathcal{I}_Y \mathcal{I}_W \subset \mathcal{I}_Y \cap \mathcal{I}_W$$
and thus $Y \cup W \subset Y + W \subset X$ as schemes. Also, $\mathrm{Supp}(Y \cup W) = \mathrm{Supp}(Y + W)$.

Theorem 4.33 (Basic properties of local heights)**.** *Let X be a projective scheme over $\overline{K}$. Let $Y, W \subset X$ be closed subschemes such that $Y \cap \mathrm{Ass}(X) = W \cap \mathrm{Ass}(X) = \varnothing$. Fix local heights λ_Y and λ_W associated with Y and W.*

(1) *Fix local heights $\lambda_{Y \cap W}$ associated with $Y \cap W$. Then*
$$\lambda_{Y \cap W} = \min\{\lambda_Y, \lambda_W\} + O_{M_K}(1)$$
on $(X \backslash (Y \cap W))(\overline{K}) \times M(\overline{K})$.

(2) *Fix local heights λ_{Y+W} associated with $Y+W$. Then*

$$\lambda_{Y+W} = \lambda_Y + \lambda_W + O_{M_K}(1)$$

on $(X\backslash(Y\cup W))(\overline{K})\times M(\overline{K})$.

(3) *If $Y \subset W$ as schemes, then*

$$\lambda_Y \leqslant \lambda_W + O_{M_K}(1)$$

on $(X\backslash Y)(\overline{K})\times M(\overline{K})$.

(4) *Fix local heights $\lambda_{Y\cup W}$ associated with $Y\cup W$. Then*

$$\begin{aligned}&\max\{\lambda_Y,\lambda_W\} \leqslant \lambda_{Y\cup W} + O_{M_K}(1);\\ &\lambda_{Y\cup W} \leqslant \lambda_Y + \lambda_W + O_{M_K}(1)\end{aligned}$$

on $(X\backslash(Y\cup W))(\overline{K})\times M(\overline{K})$.

(5) *If* $\operatorname{Supp} Y \subset \operatorname{Supp} W$, *then there exists a constant $C > 0$ such that*

$$\lambda_Y \leqslant C\lambda_W + O_{M_K}(1)$$

on $(X\backslash Y)(\overline{K})\times M(\overline{K})$.

(6) *Let $\varphi\colon X' \longrightarrow X$ be a morphism where X' is a projective scheme over $\overline{K}$. Suppose $\varphi(\mathrm{Ass}(X'))\cap Y = \varnothing$. Then we can define $\lambda_{\varphi^{-1}(Y)}$,where $\varphi^{-1}(Y)$ is the scheme theoretic preimage, and*

$$\lambda_Y\circ(\varphi\times\mathrm{id}) = \lambda_{\varphi^{-1}(Y)} + O_{M_K}(1)$$

on $(X'\backslash\varphi^{-1}(Y))(\overline{K})\times M(\overline{K})$.

Proof. This follows form Proposition 4.21 and Construction 4.29. □

5 Arithmetic distance function

In this section we fix an infinite field K equipped with a proper set of absolute values M_K. We fix an algebraic closure $\overline{K}$ of K and let

$$M = M(\overline{K}) = \left\{ |\ |_v \;\middle|\; \begin{array}{c} |\ |_v \text{ is an absolute value on } \overline{K} \text{ such that} \\ \text{the restriction } |\ |_v|_K \text{ on } K \text{ is an element of } M_K \end{array} \right\}.$$

Definition 5.1 (Arithmetic distance function)**.** Let X be a projective scheme over $\overline{K}$. Let $\Delta_X \subset X \times X$ be the diagonal. Suppose $\Delta_X \cap \mathrm{Ass}(X \times X) = \varnothing$. The local height function λ_{Δ_X} associated with Δ_X is called arithmetic distance function on X and denoted by δ_X:

$$\delta_X = \lambda_{\Delta_X} \colon (X \times X \backslash \Delta_X)(\overline{K}) \times M(\overline{K}) \longrightarrow \mathbb{R}.$$

Note that this is determined up to M_K-bounded function. We set $\delta_X(x, x, v) = \infty$ for $x \in X(\overline{K})$ and $v \in M(\overline{K})$.

Remark 5.2. If X is reduced and all the irreducible components have dim $\geqslant 1$, then $\Delta_X \cap \mathrm{Ass}(X \times X) = \varnothing$.

Proposition 5.3. *Let $K \subset F \subset \overline{K}$ be an intermediate field such that $[F : K] < \infty$. Let X be a projective scheme over F. Suppose $\Delta_X \cap \mathrm{Ass}(X \times X) = \varnothing$. Then we can take an arithmetic distance function $\delta_{X_{\overline{K}}}$ so that the following holds. For any intermediate field $F \subset L \subset \overline{K}$ with $[L : F] < \infty$, $\delta_{X_{\overline{K}}}$ induces the indicated map:*

$$\begin{tikzcd}
(X \times X \backslash \Delta_X)(L) \times M(\overline{K}) \arrow[r, hook] \arrow[d, "\mathrm{id} \times (\ )|_L"'] & (X \times X \backslash \Delta_X)(\overline{K}) \times M(\overline{K}) \arrow[r, "\delta_{X_{\overline{K}}}"] & \mathbb{R} \\
(X \times X \backslash \Delta_X)(L) \times M(L) \arrow[urr, bend right, "\exists"'] & &
\end{tikzcd}.$$

Namely, $\delta_{X_{\overline{K}}} = \lambda_{\widetilde{\Delta_X}}$ works for any presentation $\widetilde{\Delta_X}$ of Δ_X.

Proof. This follows from Proposition 4.32. □

Proposition 5.4 (Basic properties of arithmetic distance function)**.** *Let X be a projective scheme over $\overline{K}$ and suppose $\Delta_X \cap (X \times X) = \varnothing$. Fix an arithmetic distance function δ_X.*

(1) (Symmetry) *There exists an M_K-constant γ such that*

$$|\delta_X(x, y, v) - \delta_X(y, x, v)| \leqslant \gamma(v)$$

for all $(x, y, v) \in (X \times X \backslash \Delta_X)(\overline{K}) \times M(\overline{K})$.

(2) (Triangle inequality I) *There exists an M_K-constant γ such that*

$$\delta_X(x, z, v) + \gamma(v) \geqslant \min\{\delta_X(x, y, v), \delta_X(y, z, v)\}$$

for all $(x, y, z, v) \in X(\overline{K})^3 \times M(\overline{K})$.

(3) (Triangle inequality II) *Let $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Fix a local height λ_Y associated with Y. Then there is an M_K-constant γ such that*

$$\lambda_Y(y,v) + \gamma(v) \geqslant \min\{\lambda_Y(x,v), \delta_X(x,y,v)\}$$

for all $(x,y,v) \in X(\overline{K})^2 \times M(\overline{K})$.

(4) *Let $y \in X(\overline{K})$. Suppose $y \in X$, as a closed point, is not an associated point of X. Fix a local height function λ_y associated with $\{y\}$. Then there is an M_K-constant γ such that*

$$|\delta_X(x,y,v) - \lambda_y(x,v)| \leqslant \gamma(v)$$

for all $(x,v) \in (X(\overline{K})\backslash\{y\}) \times M(\overline{K})$.

Proof. (1) Apply Theorem 4.33(6) to the automorphism

$$\sigma\colon X \times X \longrightarrow X \times X; (x,y) \mapsto (y,x).$$

Note that $\sigma^{-1}(\Delta_X) = \Delta_X$.

(2) Consider the following projections:

$$\begin{tikzcd}
 & X \times X \times X \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[d, "\mathrm{pr}_{13}"'] \arrow[dr, "\mathrm{pr}_{23}"] & \\
X \times X & X \times X & X \times X.
\end{tikzcd}$$

Note that pr_{ij} are flat and hence we have $\mathrm{pr}_{ij}(\mathrm{Ass}(X \times X \times X)) \subset \mathrm{Ass}(X \times X)$. Then up to M_K-bounded functions, we have

$$\begin{aligned}
&\min\{\delta_X(x,y,v), \delta_X(y,z,v)\} \\
&= \min\{\lambda_{\Delta_X}(x,y,v), \lambda_{\Delta_X}(y,z,v)\} \\
&= \min\{\lambda_{\mathrm{pr}_{12}^{-1}(\Delta_X)}(x,y,z,v), \lambda_{\mathrm{pr}_{23}^{-1}(\Delta_X)}(x,y,z,v)\} && \text{by Theorem 4.33 (6)} \\
&= \lambda_{\mathrm{pr}_{12}^{-1}(\Delta_X) \cap \mathrm{pr}_{23}^{-1}(\Delta_X)}(x,y,z,v) && \text{by Theorem 4.33 (1)} \\
&\leqslant \lambda_{\mathrm{pr}_{13}^{-1}(\Delta_X)}(x,y,z,v) && \begin{array}{c}\mathrm{pr}_{12}^{-1}(\Delta_X) \cap \mathrm{pr}_{23}^{-1}(\Delta_X) \subset \mathrm{pr}_{13}^{-1}(\Delta_X) \\ \text{and Theorem 4.33(3)}\end{array} \\
&= \lambda_{\Delta_X}(x,z,v) = \delta_X(x,z,v).
\end{aligned}$$

(3) Consider the following diagram:

$$\begin{tikzcd} X \times X \arrow[d, "\mathrm{pr}_1"'] & \\ X \supset Y. \end{tikzcd}$$

Since pr_1 is flat, we have $\mathrm{pr}_1(\mathrm{Ass}(X \times X)) \subset \mathrm{Ass}(X)$. Then up to M_K-bounded functions, we have

$$\begin{aligned} &\min\{\lambda_Y(x,v), \delta_X(x,y,v)\} \\ &= \min\{\lambda_{\mathrm{pr}_1^{-1}(Y)}(x,y,v), \lambda_{\Delta_X}(x,y,v)\} \\ &= \lambda_{\mathrm{pr}_1^{-1}(Y)\cap\Delta_X}(x,y,v) && \text{by Theorem 4.33 (1)} \\ &\leqslant \lambda_{\mathrm{pr}_2^{-1}(Y)}(x,y,v) && \begin{array}{c}\mathrm{pr}_1^{-1}(Y)\cap\Delta_X \subset \mathrm{pr}_2^{-1}(Y) \\ \text{and Theorem 4.33(3)}\end{array} \\ &= \lambda_Y(y,v). \end{aligned}$$

(4) Consider the embedding

$$i\colon X \longrightarrow X \times X; x \mapsto (x,y).$$

Since $y \notin \mathrm{Ass}(X)$, $i(\mathrm{Ass}(X)) \cap \Delta_X = \varnothing$. Thus by Theorem 4.33 (6), up to M_K-bounded function, we have

$$\delta_X(x,y,v) = \lambda_{\Delta_X}(x,y,v) = \lambda_{i^{-1}(\Delta_X)}(x,v) = \lambda_y(x,v).$$

□

6 Local heights on quasi-projective schemes

On quasi-projective schemes, a closed subscheme does not necessarily determine a local height function up to M_K-bounded function. However, it is still possible to attach a local height function up to local height function of "boundary".

6.1 Good projectivization

In this subsection we work over a field k.

Definition 6.1. Let X be a quasi-projective scheme over k. A good projectivization of X is a projective scheme $\overline{X}$ over k and an open immersion

$$i\colon X \longrightarrow \overline{X}$$

over k such that $\mathrm{Ass}(\overline{X}) \subset X$.

Remark 6.2. Good projectivization is compatible with base change. That is, let $k \subset k'$ be arbitrary field extension. Let X be a quasi-projective scheme over k and $\overline{X}$ be a projective scheme over k. Let $i\colon X \longrightarrow \overline{X}$ be a morphism over k. Then i is a good projectivization if and only if the base change $i_{k'}\colon X_{k'} \longrightarrow \overline{X}_{k'}$ is a good projectivization. Indeed, the equivalence of being open immersion follows from fpqc descent. Let $p\colon \overline{X}_{k'} \longrightarrow \overline{X}$ be the projection. Since p is flat, we have $p(\mathrm{Ass}(\overline{X}_{k'})) = \mathrm{Ass}(\overline{X})$. Thus $\mathrm{Ass}(\overline{X}) \subset X$ if and only if $\mathrm{Ass}(\overline{X}_{k'}) \subset p^{-1}(X) = X_{k'}$.

Note that the condition $\mathrm{Ass}(\overline{X}) \subset X$ implies X is dense in $\overline{X}$. Let $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. For any good projectivization $X \subset \overline{X}$ and any closed subscheme $\widetilde{Y} \subset \overline{X}$ such that $\widetilde{Y} \cap X = Y$, we have $\widetilde{Y} \cap \mathrm{Ass}(\overline{X}) = \varnothing$.

For a given quasi-projective scheme X and an open immersion $i\colon X \longrightarrow \overline{X}$ into a projective scheme, there are two ways to modify $\overline{X}$ so that it becomes a good projectivization.

Lemma 6.3. *Let X be a quasi-projective scheme over k. Let $i\colon X \longrightarrow \overline{X}$ be an open immersion into a projective scheme $\overline{X}$ such that X is dense in $\overline{X}$.*

(1) *There is a closed subscheme $j\colon \overline{X}_0 \subset \overline{X}$ such that*

  (a) *j is the identity on the underlying topological spaces;*

  (b) *j is isomorphic on X;*

  (c) *$\mathrm{Ass}(\overline{X}_0) \subset X$.*

  In particular, $\overline{X}_0$ is a good projectivization of X.

(2) *Let $Z \subset \overline{X}$ be a closed subscheme with support $\overline{X} \backslash X$. Let $\pi\colon \overline{X}_1 \longrightarrow \overline{X}$ be the blow-up of $\overline{X}$ along Z. Then*

  (a) *$\pi^{-1}(X)$ is dense in $\overline{X}_1$ and isomorphic to X;*

  (b) *$\pi^{-1}(Z) \subset \overline{X}_1$ is an effective Cartier divisor.*

  In particular, $\overline{X}_1$ is a good projectivization of X.

Proof. (1) First take an open affine subset $U = \operatorname{Spec} A \subset \overline{X}$. Let I be the radical ideal such that $X \cap U = U \backslash V(I)$, where $V(I)$ is the set of primes containing I. Let

$$0 = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_r$$

be a minimal primary decomposition of 0. By changing labels, we may assume $I \not\subset \sqrt{\mathfrak{q}_i}$ for $i = 1, \ldots, s$ and $I \subset \sqrt{\mathfrak{q}_j}$ for $j = s+1, \ldots, r$. Then the ideal

$$\mathfrak{a} = \mathfrak{q}_1 \cap \cdots \cap \mathfrak{q}_s$$

is independent of the choice of minimal primary decomposition by the 2nd uniqueness theorem [1, Theorem 4.10]. Since X is dense in $\overline{X}$, $\sqrt{\mathfrak{q}_j}$, $j = s+1, \ldots, r$ are not minimal primes. Thus $\mathfrak{a}$ is a nilpotent ideal. Set $U_0 = \operatorname{Spec} A/\mathfrak{a}$. By the uniqueness of $\mathfrak{a}$, this construction glue together and define desired $\overline{X}_0$.

(2) This follows from basic properties of blow-ups. □

Lemma 6.4. *Let X, X' be quasi-projective schemes over k. Let*

$$i \colon X \longrightarrow \overline{X}, \quad j \colon X' \longrightarrow \overline{X}'$$

be good projectivizations. Then the product

$$i \times j \colon X \times X' \longrightarrow \overline{X} \times \overline{X}'$$

is also a good projectivization.

Proof. Since i, j are open immersions, so is $i \times j$. Since the projection $\mathrm{pr}_i, i = 1, 2$ from $\overline{X} \times \overline{X}'$ to each factor is flat, we have

$$\begin{aligned} \operatorname{Ass}(\overline{X} \times \overline{X}') &\subset \mathrm{pr}_1^{-1}(\operatorname{Ass}(\overline{X})) \cap \mathrm{pr}_2^{-1}(\operatorname{Ass}(\overline{X}')) \\ &\subset \mathrm{pr}_1^{-1}(X) \cap \mathrm{pr}_2^{-1}(X') = X \times X'. \end{aligned}$$

□

6.2 Meromorphic functions and meromorphic sections

In this subsection, we briefly recall the definition of the sheaf of meromorphic functions and meromorphic sections of invertible sheaves on Noetherian schemes (cf. [5, 7.1.1] for the sheaf of meromorphic functions).

6.2.1 The sheaf of meromorphic functions

Definition 6.5. Let X be a Noetherian scheme.

(1) For any open subset $U \subset X$, define

$$\mathcal{R}_X(U) := \left\{ a \in \mathcal{O}_X(U) \,\middle|\, \begin{array}{c}\text{the germ } a_x \in \mathcal{O}_{X,x} \text{ at any point} \\ x \in U \text{ is a non-zero divisor}\end{array} \right\}.$$

This defines a sheaf $\mathbb{R}_X$ on X.

(2) For any open subset $U \subset X$, define

$$\mathcal{K}'_X(U) := \mathcal{R}_X(U)^{-1}\mathcal{O}_X(U).$$

This defines a presheaf $\mathcal{K}'_X$ of rings on X.

(3) The sheafification of $\mathcal{K}'_X$ is denoted by $\mathcal{K}_X$ and is called the sheaf of meromorphic functions on X.

Proposition 6.6. *Let X be a Noetherian scheme.*

(1) *We have $\mathcal{O}_X \subset \mathcal{K}'_X \subset \mathcal{K}_X$ as presheaves on X.*

(2) *For any $x \in X$, we have $\mathcal{K}'_{X,x} \simeq \mathcal{K}_{X,x} \simeq \operatorname{Frac} \mathcal{O}_{X,x}$ as $\mathcal{O}_{X,x}$-algebras.*

(3) *Let $U \subset X$ be an open subset such that $\operatorname{Ass}(X) \subset U$. Let $i\colon U \longrightarrow X$ be the inclusion. Then we have*

$$\begin{aligned} \mathcal{O}_X &\longrightarrow i_*\mathcal{O}_U \text{ is injective;} \\ \mathcal{K}_X &\longrightarrow i_*\mathcal{K}_U \text{ is isomorphic.} \end{aligned}$$

In particular, we have $\mathcal{K}_X(X) \simeq \mathcal{K}_X(U)$ by the restriction.

(4) *Let $U \subset X$ be an open affine subset. Then we have*

$$\mathcal{K}_X(U) = \operatorname{Frac} \mathcal{O}_X(U).$$

Proof. See [5, 7.1.1]. □

6.2.2 Cartier divisors and meromorphic sections

Definition 6.7. Let X be a Noetherian scheme. A Cartier divisor on X is a global section of the sheaf $\mathcal{K}_X^\times/\mathcal{O}_X^\times$. Note that a Cartier divisor is represented by a set of pairs $\{(U_i, f_i)\}_i$ where $\{U_i\}_i$ is an open cover of X, $f_i \in \mathcal{K}_X^\times(U_i)$, and $f_i/f_j \in \mathcal{O}_X^\times(U_i \cap U_j)$.

Let D be a Cartier divisor on a Noetherian scheme X. Take a representation $\{(U_i, f_i)\}_i$ of D. Define

$$\mathcal{I}_D := \text{the } \mathcal{O}_X\text{-submodule of } \mathcal{K}_X \text{ generated by } f_i\text{'s}$$

and

$$\mathcal{O}_X(D) := \mathcal{H}om_{\mathcal{O}_X}(\mathcal{I}_D, \mathcal{O}_X).$$

Note that $\mathcal{I}_D$ and $\mathcal{O}_X(D)$ are invertible $\mathcal{O}_X$-modules. Note that $\mathcal{O}_X(D)$ can be embedded into $\mathcal{K}_X$ canonically by

$$\mathcal{O}_X(D)|_{U_i} \longrightarrow \mathcal{K}_X|_{U_i}, \quad \varphi \mapsto \frac{\varphi(f_i)}{f_i}.$$

Now, let $U \subset X$ be the largest open subset of X such that $\mathcal{I}_X|_U \subset \mathcal{O}_U$. Define

$$s_D \in H^0(U, \mathcal{O}_X(D)) = \mathrm{Hom}_{\mathcal{O}_U}(\mathcal{I}_D|_U, \mathcal{O}_U)$$

to be the inclusion $\mathcal{I}_D|_U \longrightarrow \mathcal{O}_U$. Note that

- $\mathrm{Ass}(X) \subset U$;
- s_D is a regular section of $\mathcal{O}_X(D)$ on U.

On the other hand, let $\mathcal{L}$ be an invertible $\mathcal{O}_X$-module on X and $U \subset X$ be an open subset such that $\mathrm{Ass}(X) \subset U$. Let $s \in H^0(U, \mathcal{L})$ be a regular section. Then we can construct a Cartier divisor in the following way. Take an open cover $\{U_i\}_i$ of X such that

$$\varphi_i \colon \mathcal{L}|_{U_i} \xrightarrow{\sim} \mathcal{O}_{U_i}.$$

Let $f_i \in \mathcal{K}_X^\times(U_i)$ be the image of $s|_{U \cap U_i}$ by

$$H^0(U \cap U_i, \mathcal{L}) \xrightarrow{\sim} H^0(U \cap U_i, \mathcal{O}_X) \subset \mathcal{K}_X(U \cap U_i) = \mathcal{K}_X(U_i).$$

Note that since s is a regular section, f_i is a unit element. The isomorphism $\varphi_j \circ \varphi_i^{-1} \colon \mathcal{O}_{U_{ij}} \longrightarrow \mathcal{O}_{U_{ij}}$, where $U_{ij} = U_i \cap U_j$, is a multiplication by a unit $u_{ij} \in \mathcal{O}_{U_{ij}}^\times(U_{ij})$. Thus we get $f_j/f_i = u_{ij}$ as elements in $\mathcal{K}_X(U_{ij})$. Thus $\{(U_i, f_i)\}_i$ defines a Cartier divisor D on X. By the constructions, we have

$$\begin{aligned}
&\mathcal{L} \simeq \mathcal{O}_X(D);\\
&H^0(U, \mathcal{L}) \simeq H^0(U, \mathcal{O}_X(D)), \quad s \longleftrightarrow s_D|_U.
\end{aligned}$$

6.3 Boundary functions

In this subsection we fix an infinite field K equipped with a proper set of absolute values M_K. We fix an algebraic closure $\overline{K}$ of K.

Definition 6.8. Let X be a quasi-projective scheme over K or $\overline{K}$. Let λ_1, λ_2, and λ_3 be maps from $X(\overline{K}) \times M(\overline{K})$ to $\mathbb{R}$.

(1) We write
$$\lambda_1 \ll \lambda_2$$
if there is a positive real number $C > 0$ such that
$$\lambda_1(x, v) \leqslant C\lambda_2(x, v)$$
for all $(x, v) \in X(\overline{K}) \times M(\overline{K})$.

(2) We write
$$\lambda_1 \gg\ll \lambda_2$$
if $\lambda_1 \ll \lambda_2$ and $\lambda_2 \ll \lambda_1$.

(3) We write
$$\lambda_1 = \lambda_2 + O(\lambda_3)$$
if
$$|\lambda_1 - \lambda_2| \ll \lambda_3.$$

Remark 6.9. We say $\lambda_1 \ll \lambda_2$ up to M_K-bounded function or write $\lambda_1 \ll \lambda_2 + O_{M_K}(1)$ if there is a positive constant $C > 0$ and an M_K-constant γ such that
$$\lambda_1(x, v) \leqslant C(\lambda_2(x, v) + \gamma(v))$$
for all $(x, v) \in X(\overline{K}) \times M(\overline{K})$. The same is applied to $\lambda_1 \gg\ll \lambda_2$.

Lemma 6.10. *Let $\varphi\colon X' \longrightarrow X$ be a proper morphism between quasi-projective schemes over $\overline{K}$. Let*
$$i\colon X \longrightarrow \overline{X}, \quad j\colon X' \longrightarrow \overline{X}'$$
be good projectivizations. Let $Z \subset \overline{X}$ and $Z' \subset \overline{X}'$ be closed subschemes such that
$$\operatorname{Supp} Z = \overline{X} \backslash X, \quad \operatorname{Supp} Z' = \overline{X}' \backslash X'.$$
Fix local height functions λ_Z and $\lambda_{Z'}$. Then we have
$$\lambda_Z \circ ((i \circ \varphi) \times \mathrm{id}) \gg\ll \lambda_{Z'} \circ (j \times \mathrm{id})$$
on $X'(\overline{K}) \times M(\overline{K})$ up to M_K-bounded function.

Proof. Let $\Gamma \subset \overline{X}' \times \overline{X}$ be the graph of φ, i.e. the scheme theoretic closure of $(j, i \circ \varphi)\colon X' \longrightarrow \overline{X}' \times \overline{X}$. Let $\Gamma_0 \subset \Gamma$ be a closed subscheme which is a good projectivization of X' constructed in Lemma 6.3 (1). Then we get the following commutative diagram:

$$\begin{tikzcd}
X' \arrow[r, "j"] \arrow[d, "\varphi"'] \arrow[rr, bend left, "k"] & \overline{X}' & \Gamma_0 \arrow[l, "p_1"'] \arrow[dl, "p_2"] \\
X \arrow[r, "i"'] & \overline{X} &
\end{tikzcd}$$

where p_i are the morphisms induced by projections and k is the open immersion. Since φ is proper, k is the base change of i along p_2. Thus

$$p_2^{-1}(Z) = \Gamma_0 \backslash X' = p_1^{-1}(Z')$$

as sets, where the last equality follows from the definition of the graph. In particular, $Z \cap p_2(\mathrm{Ass}(\Gamma_0)) = \varnothing$ and $Z' \cap p_1(\mathrm{Ass}(\Gamma_0))$. Therefore, we get

$$\begin{aligned}
\lambda_Z \circ ((i \circ \varphi) \times \mathrm{id}) &= \lambda_Z \circ ((p_2 \circ k) \times \mathrm{id}) + O_{M_K}(1) \\
&= \lambda_{p_2^{-1}(Z)} \circ (k \times \mathrm{id}) + O_{M_K}(1) \\
&\gg\ll \lambda_{p_1^{-1}(Z')} \circ (k \times \mathrm{id}) + O_{M_K}(1) \\
&= \lambda_{Z'} \circ (j \times \mathrm{id}) + O_{M_K}(1).
\end{aligned}$$

□

Corollary 6.11. *Let X be a quasi-projective scheme over $\overline{K}$. Let*

$$i\colon X \longrightarrow \overline{X}_1, \quad j\colon X \longrightarrow \overline{X}_2$$

be two good projectivizations. Let $Z_1 \subset \overline{X}_1$ and $Z_2 \subset \overline{X}_2$ be closed subschemes such that

$$\mathrm{Supp}\, Z_1 = \overline{X}_1 \backslash X, \quad \mathrm{Supp}\, Z_2 = \overline{X}_2 \backslash X.$$

Fix local height functions λ_{Z_1} and λ_{Z_2}. Then we have

$$\lambda_{Z_1} \circ (i \times \mathrm{id}) \gg\ll \lambda_{Z_2} \circ (j \times \mathrm{id})$$

on $X(\overline{K}) \times M(\overline{K})$ up to M_K-bounded function.

Proof. This follows from Lemma 6.10. □

Definition 6.12. Let X be a quasi-projective scheme over $\overline{K}$. Let us take a good projectivization $X \subset \overline{X}$. Let $Z \subset \overline{X}$ be a closed subscheme with $\operatorname{Supp} Z = \overline{X} \backslash X$. A function λ such that $\lambda \gg\ll \lambda_Z$ up to M_K-bounded function for such Z is denoted by $\lambda_{\partial X}$ and called a boundary function of X. By Corollary 6.11, if λ and λ' are two boundary functions of X, we have

$$\lambda \gg\ll \lambda'$$

up to M_K-bounded function.

Let λ_1 and λ_2 be two maps from $X(\overline{K}) \times M(\overline{K})$ to $\mathbb{R} \cup \{\infty\}$. The property

$$\lambda_1 \leqslant \lambda_2 + O(\lambda_{\partial X}) + O_{M_K}(1)$$

is independent of the choice of $\lambda_{\partial X}$. (We use the convention that ∞ is the max element in $\mathbb{R} \cup \{\infty\}$, $\infty \pm a = \infty$ for any $a \in \mathbb{R}$, $\infty + \infty = \infty$, and $\infty - \infty = 0$.) We write

$$\lambda_1 = \lambda_2 + O(\lambda_{\partial X}) + O_{M_K}(1)$$

if $\lambda_1 \leqslant \lambda_2 + O(\lambda_{\partial X}) + O_{M_K}(1)$ and $\lambda_2 \leqslant \lambda_1 + O(\lambda_{\partial X}) + O_{M_K}(1)$. In other words,

$$|\lambda_1(x, v) - \lambda_2(x, v)| \leqslant C\lambda_Z(x, v) + \gamma(v), \quad (x, v) \in X(\overline{K}) \times M(\overline{K})$$

for some $C > 0$ and some M_K-constant γ.

If $\lambda_1 = \lambda_2 + O(\lambda_{\partial X}) + O_{M_K}(1)$, we say λ_1 and λ_2 are equal up to boundary function.

Remark 6.13. Since any $\lambda_{\partial X}$ takes finite values on $X(\overline{K}) \times M(\overline{K})$, $\lambda_1 \leqslant \lambda_2 + O(\lambda_{\partial X}) + O_{M_K}(1)$ and $\lambda_1(x, v) = \infty$ imply $\lambda_2(x, v) = \infty$.

Remark 6.14. By Lemma 6.10, boundary functions are compatible with proper morphisms. That is, let $\varphi\colon X' \longrightarrow X$ be a proper morphism between quasi-projective schemes over $\overline{K}$. Then for any boundary functions $\lambda_{\partial X}$ and $\lambda_{\partial X'}$ on X, X' respectively, we have

$$\lambda_{\partial X} \circ (\varphi \times \mathrm{id}) \gg\ll \lambda_{\partial X'}$$

on $X'(\overline{K}) \times M(\overline{K})$ up to M_K-bounded function.

Remark 6.15. On a projective scheme, a boundary function is nothing but an M_K-bounded function.

Lemma 6.16. *Let X, X' be quasi-projective schemes over $\overline{K}$. Then for any boundary functions $\lambda_{\partial X}$ and $\lambda_{\partial X'}$ on X and X', we have*

$$\lambda_{\partial(X \times X')} \gg\ll \lambda_{\partial X} \circ (\mathrm{pr}_1 \times \mathrm{id}) + \lambda_{\partial X'} \circ (\mathrm{pr}_2 \times \mathrm{id})$$

up to M_K-bounded function. Here pr_i is the projection from $X \times X'$ to each factor.

Proof. Let

$$i\colon X \longrightarrow \overline{X}, \quad j\colon X' \longrightarrow \overline{X}'$$

be good projectivizations. Then the product

$$i \times j\colon X \times X' \longrightarrow \overline{X} \times \overline{X}'$$

is a good projectivization by Lemma 6.4. Let $Z = (\overline{X} \backslash X)_{\mathrm{red}}$ and $Z' = (\overline{X}' \backslash X')_{\mathrm{red}}$. Then $(\overline{X} \times \overline{X}') \backslash (X \times X') = \mathrm{pr}_1^{-1}(Z) \cup \mathrm{pr}_2^{-1}(Z')$ as sets. Thus

$$\lambda_{((\overline{X} \times \overline{X}') \backslash (X \times X'))_{\mathrm{red}}} \gg\ll \lambda_Z \circ (\mathrm{pr}_1 \times \mathrm{id}) + \lambda_{Z'} \circ (\mathrm{pr}_2 \times \mathrm{id})$$

up to M_K-bounded function. □

6.4 Local height functions on quasi-projective schemes

In this subsection we fix an infinite field K equipped with a proper set of absolute values M_K. We fix an algebraic closure $\overline{K}$ of K.

We will attach local height functions to closed subschemes of quasi-projective schemes up to boundary functions. On a projective scheme, local heights associated with presentations of a closed subscheme are the same up to M_K-bounded function. On a quasi-projective scheme, it is generalized to the following theorem, namely the same statement is true if we replace M_K-bounded function with boundary function.

Theorem 6.17. *Let $K \subset F \subset L \subset \overline{K}$ be intermediate fields such that $[L : K] < \infty$. Let X be a quasi-projective scheme over F. Let $Y \subset X_L$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X_L) = \varnothing$. Let $\mathcal{Y}$ and $\mathcal{Y}'$ be two presentations of Y. Then $\lambda_{\mathcal{Y}}$ and $\lambda_{\mathcal{Y}'}$ are equal up to boundary function, i.e.*

$$\lambda_{\mathcal{Y}} = \lambda_{\mathcal{Y}'} + O(\lambda_{\partial X_{\overline{K}}}) + O_{M_K}(1)$$

on $X(\overline{K}) \times M(\overline{K})$.

Corollary 6.18. *Let $\varphi\colon X' \longrightarrow X$ be a morphism between quasi-projective schemes over $\overline{K}$. Then*

$$\lambda_{\partial X} \circ (\varphi \times \mathrm{id}) = O(\lambda_{\partial X'}) + O_{M_K}(1).$$

Proof. Let $K \subset F \subset \overline{K}$ be an intermediate field such that $[F : K] < \infty$ and X, X', and φ are defined over F. Let us denote models over F by X_F, X'_F, and φ_F. (Note that we take F large enough so that X_F and X'_F are quasi-projective.) Take a good projectivization $i\colon X_F \longrightarrow \overline{X}$ over F. Let $Z = \overline{X} \backslash X_F$ with reduced structure. Take a presentation $\mathcal{Z}$ of Z. Then $i^*\mathcal{Z}$ is a presentation of the empty subscheme $\varnothing \subset X_F$. It is enough to show the statement for $\lambda_{\partial X} = \lambda_{\mathcal{Z}}$. By the functoriality, we have

$$\lambda_{\mathcal{Z}} \circ (\varphi \times \mathrm{id}) = \lambda_{\varphi^*\mathcal{Z}}.$$

Note that $\varphi^*\mathcal{Z}$ is a presentation of $\varnothing \subset X'_F$. By Theorem 6.17, we get $\lambda_{\varphi^*\mathcal{Z}} = O(\lambda_{\partial X'})$.

□

Definition 6.19 (Local heights on quasi-projective schemes)**.** Let X be a quasi-projective scheme over $\overline{K}$. Let $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Let $K \subset F \subset \overline{K}$ be an intermediate field such that $[F : K] < \infty$ and there exist a quasi-projective scheme X_F over F and a closed subscheme $Y_F \subset X_F$ such that

$$\begin{tikzcd} (X_F)_{\overline{K}} \arrow[r, "\sim"] & X \\ (Y_F)_{\overline{K}} \arrow[u, hook] \arrow[r, "\sim"] & Y \arrow[u, hook] \end{tikzcd}$$

commutes.

Let $\mathcal{Y}$ be a presentation of Y_F. Any map $X(\overline{K}) \times M(\overline{K}) \longrightarrow \mathbb{R} \cup \{\infty\}$ which is equal to $\lambda_{\mathcal{Y}}$ up to boundary function is called a height function associated with Y and denoted by λ_Y. This function λ_Y is determined up to boundary function and the definition is independent of the choice of F, X_F, and the presentation $\mathcal{Y}$.

To prove Theorem 6.17, we start with the following lemma, by which we can actually give an alternative definition of local height on quasi-projective schemes.

Lemma 6.20. *Let X be a quasi-projective scheme over $\overline{K}$. Let $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Let*

$$i\colon X \longrightarrow \overline{X}_1, \quad j\colon X \longrightarrow \overline{X}_2$$

be two good projectivizations. Let $\widetilde{Y}_1 \subset \overline{X}_1$ and $\widetilde{Y}_2 \subset \overline{X}_2$ be closed subschemes such that

$$\widetilde{Y}_1 \cap X = Y, \quad \widetilde{Y}_2 \cap X = Y.$$

Fix local heights $\lambda_{\widetilde{Y}_1}$ and $\lambda_{\widetilde{Y}_2}$. Then we have

$$\lambda_{\widetilde{Y}_1} \circ (i \times \mathrm{id}) = \lambda_{\widetilde{Y}_2} \circ (j \times \mathrm{id}) + O(\lambda_{\partial X}) + O_{M_K}(1)$$

on $X(\overline{K}) \times M(\overline{K})$.

Proof. Let $\Gamma \subset \overline{X}_1 \times \overline{X}_2$ be the scheme theoretic closure of $(i, j \circ \varphi) \colon X \longrightarrow \overline{X}_1 \times \overline{X}_2$. Let $\Gamma_0 \subset \Gamma$ be a closed subscheme which is a good projectivization of X constructed in Lemma 6.3 (1). Then we get the following commutative diagram:

$$\begin{tikzcd} X \arrow[r, "i"] \arrow[rr, bend left, "k"] \arrow[dr, "j"'] & \overline{X}_1 & \Gamma_0 \arrow[l, "p_1"'] \arrow[dl, "p_2"] \\ & \overline{X}_2 & \end{tikzcd}$$

where p_i are the morphisms induced by projections and k is the open immersion. Then we have

$$p_1^{-1}(\widetilde{Y}_1) \cap X = Y = p_2^{-1}(\widetilde{Y}_2) \cap X$$

as closed subschemes.

Claim 6.21. *Let V be an algebraic scheme, $U \subset V$ an open subset, and $Z = V \backslash U$ equipped with the reduced structure. Let W_1, W_2 be closed subschemes of V such that $W_1 \cap U = W_2 \cap U$ as closed subschemes. Then there is a non-negative integer n such that*

$$W_1 \subset W_2 + nZ.$$

Proof. Let $\mathcal{I}_{W_1}, \mathcal{I}_{W_2}$, and $\mathcal{I}_Z$ be the ideal sheaf of W_1, W_2, and Z. Let

$$\mathcal{F} = \mathrm{Im}(\mathcal{I}_{W_2} \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V/\mathcal{I}_{W_1}).$$

Then the support of this coherent sheaf $\mathcal{F}$ is contained in Z. Thus there is $n \geqslant 0$ such that $\mathcal{I}_Z^n \mathcal{F} = 0$. This implies the map

$$\mathcal{I}_Z^n \mathcal{I}_{W_2} \longrightarrow \mathcal{O}_V \longrightarrow \mathcal{O}_V/\mathcal{I}_{W_1}$$

is zero. This is what we wanted. ■

Let $Z = \Gamma_0 \backslash X$ with reduced structure. By the claim, there is $n \geqslant 0$ such that

$$p_1^{-1}(\widetilde{Y}_1) \subset p_2^{-1}(\widetilde{Y}_2) + nZ.$$

Note that for $l = 1, 2$ we have

$$\widetilde{Y}_l \cap \mathrm{Ass}(\overline{X}_l) = \widetilde{Y}_l \cap \mathrm{Ass}(X) = \varnothing$$
$$\widetilde{Y}_l \cap p_l(\mathrm{Ass}(\Gamma_0)) = \widetilde{Y}_l \cap p_l(\mathrm{Ass}(X)) = \varnothing$$

and therefore $\lambda_{\widetilde{Y}_l}$ and $\lambda_{p_l^{-1}(\widetilde{Y}_l)}$ are well-defined.

Then we get

$$\begin{aligned}\lambda_{\widetilde{Y}_1} \circ (i \times \mathrm{id}) &= \lambda_{\widetilde{Y}_1} \circ (p_1 \times \mathrm{id}) \circ (k \times \mathrm{id}) \\ &= \lambda_{p_1^{-1}(\widetilde{Y}_1)} \circ (k \times \mathrm{id}) + O_{M_K}(1) \\ &\leqslant \lambda_{p_2^{-1}(\widetilde{Y}_2)} \circ (k \times \mathrm{id}) + n\lambda_Z \circ (k \times \mathrm{id}) + O_{M_K}(1) \\ &= \lambda_{\widetilde{Y}_2} \circ (j \times \mathrm{id}) + n\lambda_Z \circ (k \times \mathrm{id}) + O_{M_K}(1)\end{aligned}$$

We can get the opposite inequality by switching the role of $\widetilde{Y}_1$ and $\widetilde{Y}_2$ and we are done.

□

Now we prove Theorem 6.17.

Proof of Theorem 6.17.

Claim 6.22. *Let X be a quasi-projective scheme over F and D be an effective Cartier divisor on X. Let $\mathcal{D}$ be a presentation of D. Then there exists*

- *a good projectivization $X \subset \overline{X}$ such that $\overline{X} \backslash X$ is the support of an effective Cartier divisor on $\overline{X}$;*
- *an effective Cartier divisor $\widetilde{D}$ on $\overline{X}$ such that $\widetilde{D}|_X = D$;*
- *a presentation $\widetilde{\mathcal{D}}$ of $\widetilde{D}$ and a presentation $\mathcal{E}$ of an effective Cartier divisor whose support is $\overline{X} \backslash X$,*

such that

$$\widetilde{\mathcal{D}}|_X = \mathcal{D} + \mathcal{E}|_X.$$

Proof. Let

$$\mathcal{D} = (s_D; \mathcal{L}, s_0, \ldots, s_n; \mathcal{M}, t_0, \ldots, t_m; \psi).$$

Since s_i's and t_j's are generating sections, they define a morphism from X to projective spaces $\mathbb{P}^n$ and $\mathbb{P}^m$:

$$\Psi_{\mathcal{L}} \colon X \longrightarrow \mathbb{P}^n = \mathrm{Proj}\, F[x_0, \ldots, x_n] =: P_1$$
$$\Psi_{\mathcal{M}} \colon X \longrightarrow \mathbb{P}^m = \mathrm{Proj}\, F[y_0, \ldots, y_m] =: P_2.$$

Let take an immersion
$$\alpha\colon X \longrightarrow \mathbb{P}^N =: P_3$$
into some projective space $\mathbb{P}^N$. Let $\overline{X}' \subset P_1 \times P_2 \times P_3$ be the scheme theoretic closure of the immersion $(\Psi_{\mathcal{L}}, \Psi_M, \alpha)\colon X \longrightarrow P_1 \times P_2 \times P_3$. Let $\overline{X}$ be the blow up of $\overline{X}'$ along $(\overline{X}'\backslash X)_{\mathrm{red}}$. We get the following diagram:

$$\begin{tikzcd} X \arrow[rrr, "(\Psi_{\mathcal{L}},\Psi_M,\alpha)"] \arrow[dr, "i"'] & & & P_1 \times P_2 \times P_3 \\ & \overline{X} \arrow[r] & \overline{X}' \arrow[ur] & \end{tikzcd}$$

where i is the open immersion and it is a good projectivization by Lemma 6.3. We identify X with its images via this diagram. Let $E \subset \overline{X}$ be the exceptional divisor of the blow up $\overline{X} \longrightarrow \overline{X}'$. Note that $E = \overline{X}\backslash X$ as sets.

Let
$$p_i\colon \overline{X} \longrightarrow P_1 \times P_2 \times P_3 \longrightarrow P_i$$
be the morphism induced by the projections. Let
$$\begin{aligned} \widetilde{\mathcal{L}} &= p_1^*\mathcal{O}_{P_1}(1), \quad \widetilde{s_0} = p_1^* x_0, \ldots, \widetilde{s_n} = p_1^* x_n \\ \widetilde{\mathcal{M}} &= p_2^*\mathcal{O}_{P_2}(1), \quad \widetilde{t_0} = p_2^* y_0, \ldots, \widetilde{t_m} = p_2^* y_m. \end{aligned}$$

By construction, we have
$$\begin{aligned} \widetilde{\mathcal{L}}|_X &= \mathcal{L}, \quad \widetilde{s_i}|_X = s_i \\ \widetilde{\mathcal{M}}|_X &= \mathcal{M}, \quad \widetilde{t_j}|_X = t_j. \end{aligned}$$

Since $\mathrm{Ass}(\overline{X}) = \mathrm{Ass}(X)$, $\widetilde{s_i}$ and $\widetilde{t_j}$ are regular sections.

By the isomorphism
$$\widetilde{\mathcal{L}} \otimes \widetilde{\mathcal{M}}^{-1}|_X \xrightarrow{\sim} \mathcal{L} \otimes \mathcal{M}^{-1} \xrightarrow[\psi]{\sim} \mathcal{O}_X(D)$$
the regular section $s_D \in H^0(X, \mathcal{O}_X(D))$ can be regraded as a regular section
$$s_D \in H^0(X, \widetilde{\mathcal{L}} \otimes \widetilde{\mathcal{M}}^{-1}).$$

Since $\mathrm{Ass}(\overline{X}) \subset X$, this is a meromorphic section. Thus by section 6.2.2, this defines a Cartier divisor D' on $\overline{X}$ and by construction
$$D'|_X = D.$$

Recall that $\overline{X}\backslash X$ is the support of the exceptional divisor E. Thus there is a non-negative integer $d \geqslant 0$ such that

$$\widetilde{D} := D' + dE$$

is effective. Note that $\widetilde{D}|_X = D$ and

$$\mathcal{O}_{\overline{X}}(\widetilde{D}) \simeq \mathcal{O}_{\overline{X}}(D') \otimes \mathcal{O}_{\overline{X}}(dE) \simeq \widetilde{\mathcal{L}} \otimes \widetilde{\mathcal{M}}^{-1} \otimes \mathcal{O}_{\overline{X}}(dE).$$

We denote this isomorphism by χ (from right to left). Take a presentation of dE:

$$\mathcal{E} = (s_{dE}; \mathcal{L}'; s'_0, \ldots, s'_{n'}; \mathcal{M}', t'_0, \ldots, t'_{m'}; \psi').$$

Then

$$\widetilde{\mathcal{D}} := (s_{\widetilde{D}}; \widetilde{\mathcal{L}} \otimes \mathcal{L}', \{\widetilde{s}_i \otimes s'_{i'};\}; \widetilde{\mathcal{M}} \otimes \mathcal{M}', \{\widetilde{t}_j \otimes t'_{j'}\}; \chi \circ \mathrm{id} \otimes \psi')$$

is a presentation of $\widetilde{D}$. Then we get

$$\widetilde{\mathcal{D}}|_X = \mathcal{D} + \mathcal{E}|_X$$

and we are done. ■

Claim 6.23. *Let X be a quasi-projective scheme over F and $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Let*

$$\mathcal{Y} = (Y; \mathcal{D}_1, \ldots, \mathcal{D}_r)$$

be a presentation of Y. Then there exist

(1) *a good projectivization $i\colon X \longrightarrow \overline{X}$;*

(2) *a closed subscheme $\widetilde{Y} \subset \overline{X}$ such that $\widetilde{Y} \cap X = Y$ (in particular, $\widetilde{Y} \cap \mathrm{Ass}(\overline{X}) = \varnothing$);*

(3) *a presentation $\widetilde{\mathcal{Y}}$ of $\widetilde{Y}$;*

(4) *presentations $\mathcal{E}_1, \ldots, \mathcal{E}_r$ of effective Cartier divisors on $\overline{X}$ whose supports are contained in $\overline{X}\backslash X$,*

such that

$$\widetilde{\mathcal{Y}}|_X = (Y; \mathcal{D}_1 + \mathcal{E}_1|_X, \ldots, \mathcal{D}_r + \mathcal{E}_r|_X).$$

Proof. Let D_i be the divisors presented by $\mathcal{D}_i$. By the definition of presentation, $Y = D_1 \cap \cdots \cap D_r$.

For each $\mathcal{D}_i$, apply Claim 6.22 and get:

- a good projectivization $j_i\colon X \longrightarrow \overline{X}_i$;
- an effective Cartier divisor $\widetilde{D}_i \subset \overline{X}_i$ and its presentation $\widetilde{\mathcal{D}_i}$;
- a presentation $\mathcal{E}_i$ of an effective Cartier divisor $E_i \subset \overline{X}_i$ such that $\operatorname{Supp} E_i = \overline{X}_i \backslash X$,

satisfying

$$\begin{aligned} j_i^* \widetilde{D}_i &= D_i \\ j_i^* \widetilde{\mathcal{D}_i} &= \mathcal{D}_i + j_i^* \mathcal{E}_i. \end{aligned}$$

Let $\overline{X}'$ be the scheme theoretic image of the morphism $(j_i)_{i=1}^r\colon X \longrightarrow \prod_{i=1}^r \overline{X}_i$. Let $\overline{X} \subset \overline{X}'$ be the closed subscheme which is a good projectivization of X constructed in Lemma 6.3 (1). We get the following diagram:

$$\begin{tikzcd}
X \arrow[rrr, "(j_i)_{i=1}^r"] \arrow[dr, "i"'] & & & \prod_{i=1}^r \overline{X}_i \arrow[r, "\mathrm{pr}_i"] & \overline{X}_i \\
& \overline{X} \arrow[r] \arrow[urr, bend right, "\alpha"'] \arrow[urrr, bend right, "p_i"'] & \overline{X}' \arrow[ur] & &
\end{tikzcd}$$

where α is the induced closed immersion, pr_i is the i-th projection, $p_i = \mathrm{pr}_i \circ \alpha$, and i is the good projectivization. Note that $p_i(\operatorname{Ass}(\overline{X})) = p_i(\operatorname{Ass}(X)) = j_i(\operatorname{Ass}(X)) = \operatorname{Ass}(\overline{X}_i)$. Therefore we can pull-back presentations on $\overline{X}_i$ by p_i.

Let
$$\widetilde{Y} = p_1^* \widetilde{D}_1 \cap \cdots \cap p_r^* \widetilde{D}_r.$$
Then $\widetilde{Y} \cap \operatorname{Ass}(\overline{X}) = \varnothing$ and $\widetilde{Y} \cap X = Y$. Let

$$\widetilde{\mathcal{Y}} = (\widetilde{Y}; p_1^* \widetilde{\mathcal{D}_1}, \ldots, p_r^* \widetilde{\mathcal{D}_r}).$$

This is a presentation of $\widetilde{Y}$ and

$$\widetilde{\mathcal{Y}}|_X = (Y; \mathcal{D}_1 + p_1^* \mathcal{E}_1|_X, \ldots, \mathcal{D}_r + p_r^* \mathcal{E}_r|_X).$$

Note that $p_i^* \mathcal{E}_i$ is a presentation of $p_i^* E_i$ and $\operatorname{Supp} p_i^* E_i \subset \overline{X} \backslash X$. ∎

Now let X be a quasi-projective scheme over F and $Y \subset X_L$ be a closed subscheme such that $Y \cap \operatorname{Ass}(X_L) = \varnothing$. Let $\mathcal{Y} = (Y; \mathcal{D}_1, \ldots, \mathcal{D}_r)$ be a presentation of Y. Take $i\colon X_L \to \overline{X}, \widetilde{Y}, \widetilde{\mathcal{Y}}$, and $\mathcal{E}_i$ as in Claim 6.23 (apply on X_L and set $F = L$). Then on $X(\overline{K}) \times M(\overline{K})$, we have

$$\lambda_{\mathcal{Y}} = \min\{\lambda_{\mathcal{D}_1}, \dots, \lambda_{\mathcal{D}_r}\}$$

and

$$\lambda_{\widetilde{\mathcal{Y}}} \circ (i \times \mathrm{id}) = \min\{\lambda_{\mathcal{D}_1} + \lambda_{\mathcal{E}_1} \circ (i \times \mathrm{id}), \dots, \lambda_{\mathcal{D}_r} + \lambda_{\mathcal{E}_r} \circ (i \times \mathrm{id})\}.$$

Thus

$$\begin{aligned} \lambda_{\widetilde{\mathcal{Y}}} \circ (i \times \mathrm{id}) &\leqslant \lambda_{\mathcal{Y}} + \max\{\lambda_{\mathcal{E}_1} \circ (i \times \mathrm{id}), \dots, \lambda_{\mathcal{E}_r} \circ (i \times \mathrm{id})\} \\ \lambda_{\widetilde{\mathcal{Y}}} \circ (i \times \mathrm{id}) &\geqslant \lambda_{\mathcal{Y}} + \min\{\lambda_{\mathcal{E}_1} \circ (i \times \mathrm{id}), \dots, \lambda_{\mathcal{E}_r} \circ (i \times \mathrm{id})\}. \end{aligned}$$

Since $\mathcal{E}_i$ are presentations of divisors supported in $\overline{X} \backslash X_L$, we have

$$\begin{aligned} \max\{\lambda_{\mathcal{E}_1} \circ (i \times \mathrm{id}), \dots, \lambda_{\mathcal{E}_r} \circ (i \times \mathrm{id})\} &= O(\lambda_{\partial X}) + O_{M_K}(1) \\ \min\{\lambda_{\mathcal{E}_1} \circ (i \times \mathrm{id}), \dots, \lambda_{\mathcal{E}_r} \circ (i \times \mathrm{id})\} &= O(\lambda_{\partial X}) + O_{M_K}(1). \end{aligned}$$

This implies

$$\lambda_{\mathcal{Y}} = \lambda_{\widetilde{\mathcal{Y}}} \circ (i \times \mathrm{id}) + O(\lambda_{\partial X}) + O_{M_K}(1).$$

Note that $\lambda_{\widetilde{\mathcal{Y}}}$ is a local height on a projective scheme $\overline{X}$. Hence the statement follows from Lemma 6.20.

□

Theorem 6.24 (Basic properties of local heights on quasi-projective schemes)**.** *Let X be a quasi-projective scheme over $\overline{K}$. Let $Y, W \subset X$ be closed subschemes such that $Y \cap \mathrm{Ass}(X) = W \cap \mathrm{Ass}(X) = \varnothing$. Fix local heights λ_Y and λ_W associated with Y and W.*

(1) *Fix local heights $\lambda_{Y \cap W}$ associated with $Y \cap W$. Then*

$$\lambda_{Y \cap W} = \min\{\lambda_Y, \lambda_W\} + O(\lambda_{\partial X}) + O_{M_K}(1)$$

on $X(\overline{K}) \times M(\overline{K})$.

(2) *Fix local heights λ_{Y+W} associated with $Y + W$. Then*

$$\lambda_{Y+W} = \lambda_Y + \lambda_W + O(\lambda_{\partial X}) + O_{M_K}(1)$$

on $X(\overline{K}) \times M(\overline{K})$.

(3) *If $Y \subset W$ as schemes, then*

$$\lambda_Y \leqslant \lambda_W + O(\lambda_{\partial X}) + O_{M_K}(1)$$

on $X(\overline{K}) \times M(\overline{K})$.

(4) *Fix local heights $\lambda_{Y\cup W}$ associated with $Y \cup W$. Then*

$$\max\{\lambda_Y, \lambda_W\} \leqslant \lambda_{Y\cup W} + O(\lambda_{\partial X}) + O_{M_K}(1);$$
$$\lambda_{Y\cup W} \leqslant \lambda_Y + \lambda_W + O(\lambda_{\partial X}) + O_{M_K}(1)$$

on $X(\overline{K}) \times M(\overline{K})$.

(5) *If* $\operatorname{Supp} Y \subset \operatorname{Supp} W$*, then there exists a constant $C > 0$ such that*

$$\lambda_Y \leqslant C\lambda_W + O(\lambda_{\partial X}) + O_{M_K}(1)$$

on $X(\overline{K}) \times M(\overline{K})$.

(6) *Let $\varphi\colon X' \longrightarrow X$ be a morphism where X' is a quasi-projective scheme over $\overline{K}$. Suppose $\varphi(\operatorname{Ass}(X')) \cap Y = \varnothing$. Then we can define $\lambda_{\varphi^{-1}(Y)}$,where $\varphi^{-1}(Y)$ is the scheme theoretic preimage, and*

$$\lambda_Y \circ (\varphi \times \mathrm{id}) = \lambda_{\varphi^{-1}(Y)} + O(\lambda_{\partial X'}) + O_{M_K}(1)$$

on $X'(\overline{K}) \times M(\overline{K})$.

(7) *Let $\varphi\colon X' \longrightarrow X$ be a proper morphism where X' is a quasi-projective scheme over $\overline{K}$. Then for any boundary functions $\lambda_{\partial X}$ and $\lambda_{\partial X'}$, we have*

$$\lambda_{\partial X} \circ (\varphi \times \mathrm{id}) \gg\ll \lambda_{\partial X'}$$

on $X'(\overline{K}) \times M(\overline{K})$ up to M_K-bounded function.

Proof. Follows from the definition of local height associated with presentations and Proposition 4.21, Corollary 6.18, Theorem 6.17, and Remark 6.14. □

6.5 Arithmetic distance function on quasi-projective schemes

In this subsection we fix an infinite field K equipped with a proper set of absolute values M_K. We fix an algebraic closure $\overline{K}$ of K.

Definition 6.25. Let X a quasi-projective scheme over $\overline{K}$. Suppose $\Delta_X \cap \operatorname{Ass}(X \times X) = \varnothing$. Then a local height function associated with Δ_X is denoted by δ_X and called an arithmetic distance function on X:

$$\delta_X(x, y, v) = \lambda_{\Delta_X}(x, y, v)$$

for $(x, y, v) \in (X \times X)(\overline{K}) \times M(\overline{K})$. Note that this is determined up to boundary function on $X \times X$.

Proposition 6.26 (Basic properties of arithmetic distance function on quasi-projective schemes)**.** *Let X be a quasi-projective scheme over $\overline{K}$ and suppose $\Delta_X \cap (X \times X) = \varnothing$. Fix an arithmetic distance function δ_X.*

(1) (Symmetry) *We have*

$$\delta_X(x, y, v) = \delta_X(y, x, v) + O(\lambda_{\partial(X\times X)}(x, y, v)) + O_{M_K}(1)$$

for all $(x, y, v) \in (X \times X)(\overline{K}) \times M(\overline{K})$.

(2) (Triangle inequality I) *We have*

$$\min\{\delta_X(x, y, v), \delta_X(y, z, v)\} \leqslant \delta_X(x, z, v) + O(\lambda_{\partial(X\times X\times X)}(x, y, z, v)) + O_{M_K}(1)$$

for all $(x, y, z, v) \in X(\overline{K})^3 \times M(\overline{K})$.

(3) (Triangle inequality II) *Let $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Fix a local height λ_Y associated with Y. Then we have*

$$\min\{\lambda_Y(x, v), \delta_X(x, y, v)\} \leqslant \lambda_Y(y, v) + O(\lambda_{\partial(X\times X)}(x, y, v)) + O_{M_K}(1)$$

for all $(x, y, v) \in X(\overline{K})^2 \times M(\overline{K})$.

(4) *Let $y \in X(\overline{K})$. Suppose $y \in X$, as a closed point, is not an associated point of X. Fix a local height function λ_y associated with $\{y\}$. Then*

$$\delta_X(x, y, v) = \lambda_y(x, v) + O(\lambda_{\partial X}(x, v)) + O_{M_K}(1)$$

for all $(x, v) \in (X(\overline{K})\backslash\{y\}) \times M(\overline{K})$.

Proof. (1) Apply Theorem 6.24(6) to the automorphism

$$\sigma\colon X \times X \longrightarrow X \times X; (x, y) \mapsto (y, x).$$

Note that $\sigma^{-1}(\Delta_X) = \Delta_X$.

(2) Consider the following projections:

$$\begin{tikzcd} & X \times X \times X \arrow[dl, "\mathrm{pr}_{12}"'] \arrow[d, "\mathrm{pr}_{13}"] \arrow[dr, "\mathrm{pr}_{23}"] & \\ X \times X & X \times X & X \times X. \end{tikzcd}$$

Note that pr_{ij} are flat and hence we have $\mathrm{pr}_{ij}(\mathrm{Ass}(X\times X\times X)) \subset \mathrm{Ass}(X\times X)$. Then up to M_K-bounded functions, we have

$$\min\{\delta_X(x, y, v), \delta_X(y, z, v)\}$$

$$= \min\{\lambda_{\Delta_X}(x,y,v) + O(\lambda_{\partial(X\times X)}(x,y,v)), \lambda_{\Delta_X}(y,z,v) + O(\lambda_{\partial(X\times X)}(y,z,v))\}$$

$$= \min\{\lambda_{\mathrm{pr}_{12}^{-1}(\Delta_X)}(x,y,z,v), \lambda_{\mathrm{pr}_{23}^{-1}(\Delta_X)}(x,y,z,v)\} + O(\lambda_{\partial(X\times X\times X)}(x,y,z,v))$$

by Theorem 6.24 (6)

$$= \lambda_{\mathrm{pr}_{12}^{-1}(\Delta_X)\cap\mathrm{pr}_{23}^{-1}(\Delta_X)}(x,y,z,v) + O(\lambda_{\partial(X\times X\times X)}(x,y,z,v))$$

by Theorem 6.24 (1)

$$\leqslant \lambda_{\mathrm{pr}_{13}^{-1}(\Delta_X)}(x,y,z,v) + O(\lambda_{\partial(X\times X\times X)}(x,y,z,v))$$

$\mathrm{pr}_{12}^{-1}(\Delta_X)\cap\mathrm{pr}_{23}^{-1}(\Delta_X)\subset\mathrm{pr}_{13}^{-1}(\Delta_X)$
and Theorem 6.24(3)

$$= \lambda_{\Delta_X}(x,z,v) + O(\lambda_{\partial(X\times X\times X)}(x,y,z,v))$$

$$= \delta_X(x,z,v) + O(\lambda_{\partial(X\times X\times X)}(x,y,z,v)).$$

(3) Consider the following diagram:

$$\begin{array}{c} X\times X \\ \Big\downarrow{\scriptstyle \mathrm{pr}_1} \\ X\supset Y. \end{array}$$

Since pr_1 is flat, we have $\mathrm{pr}_1(\mathrm{Ass}(X\times X))\subset\mathrm{Ass}(X)$. Then up to M_K-bounded functions, we have

$$\min\{\lambda_Y(x,v), \delta_X(x,y,v)\}$$

$$= \min\{\lambda_{\mathrm{pr}_1^{-1}(Y)}(x,y,v), \lambda_{\Delta_X}(x,y,v)\} + O(\lambda_{\partial(X\times X)}(x,y,v))$$

$$= \lambda_{\mathrm{pr}_1^{-1}(Y)\cap\Delta_X}(x,y,v) + O(\lambda_{\partial(X\times X)}(x,y,v)) \qquad \text{by Theorem 6.24 (1)}$$

$$\leqslant \lambda_{\mathrm{pr}_2^{-1}(Y)}(x,y,v) + O(\lambda_{\partial(X\times X)}(x,y,v))$$

$\mathrm{pr}_1^{-1}(Y)\cap\Delta_X\subset\mathrm{pr}_2^{-1}(Y)$
and Theorem 6.24(3)

$$= \lambda_Y(y,v) + O(\lambda_{\partial(X\times X)}(x,y,v)).$$

(4) Consider the embedding

$$i\colon X\longrightarrow X\times X; x\mapsto(x,y).$$

Since $y\notin\mathrm{Ass}(X)$, $i(\mathrm{Ass}(X))\cap\Delta_X=\varnothing$. Thus by Theorem 6.24 (6), up to M_K-bounded functions, we have

$$\delta_X(x,y,v) = \lambda_{\Delta_X}(x,y,v) + O(\lambda_{\partial(X\times X)}(x,y,v))$$

$$\begin{aligned}&= \lambda_{i^{-1}(\Delta_X)}(x, v) + O(\lambda_{\partial(X\times X)}(x, y, v))\\&= \lambda_y(x, v) + O(\lambda_{\partial(X\times X)}(x, y, v)).\end{aligned}$$

Here $\lambda_{\partial(X\times X)}(x, y, v) = \lambda_{\partial(X\times X)} \circ (i \times \mathrm{id})(x, v) = O(\lambda_{\partial X}(x, v))$ and we are done.

□

7 Global heights

In this section we fix an infinite field K equipped with a proper set of absolute values M_K. We fix an algebraic closure $\overline{K}$ of K. Recall that for an intermediate field $K \subset L \subset \overline{K}$, $M(L)$ is the set of absolute values on L which extend elements of M_K.

Construction 7.1 (Global height function associated with a presentation)**.** Let $K \subset L \subset \overline{K}$ be an intermediate field such that $[L : K] < \infty$. Let X be a projective scheme over L and $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Let $\mathcal{Y}$ be a presentation of Y:

$$\begin{aligned}&\mathcal{Y} = (Y_L; \mathcal{D}_1, \ldots, \mathcal{D}_r) \qquad \text{where}\\&\mathcal{D}_i = (s_{D_i}; \mathcal{L}^{(i)}, s_0^{(i)}, \ldots, s_{n_i}^{(i)}; \mathcal{M}^{(i)}, t_0^{(i)}, \ldots, t_{m_i}^{(i)}).\end{aligned}$$

For any intermediate field $L \subset L' \subset \overline{K}$, define

$$\lambda_{\mathcal{Y},L'} \colon (X\backslash Y)(L') \times M(L') \longrightarrow \mathbb{R}$$

by

$$\lambda_{\mathcal{Y},L'}(x, v) = \min\{\lambda_{\mathcal{D}_1,L'}(x, v), \ldots, \lambda_{\mathcal{D}_r,L'}(x, v)\}$$

where

$$\lambda_{\mathcal{D}_i,L'}(x, v) = \log \max_{0\leqslant k\leqslant n_i} \min_{0\leqslant l\leqslant m_i} \left\{ \left| \frac{s_k^{(i)}}{s_{D_i} t_l^{(i)}}(x) \right|_v \right\}$$

for $(x, v) \in (X\backslash Y)(L') \times M(L')$. Note that this map is the induced map in Proposition 4.32.

Now define

$$h_{\mathcal{Y}} \colon (X\backslash Y)(\overline{K}) \longrightarrow \mathbb{R}$$

as follows. For a point $x \in (X\backslash Y)(\overline{K})$, take an intermediate field $L \subset L' \subset \overline{K}$ such that $[L' : L] < \infty$ and $x \in (X\backslash Y)(L')$. Then

$$h_{\mathcal{Y}}(x) = \frac{1}{[L' : K]} \sum_{v\in M(L')} [L'_v : K_{v|_K}] \lambda_{\mathcal{Y},L'}(x, v).$$

Claim 7.2. *This map is well-defined, that is, $h_{\mathcal{Y}}(x)$ is independent of the choice of L'.*

Proof. Let L' and L'' be two intermediate fields which are finite over L and such that $x \in (X\backslash Y)(L')$ and $x \in (X\backslash Y)(L'')$. We prove $h_{\mathcal{Y}}(x)$'s defined by L' and L'' are the same. By replacing L'' with the composite field $L' \cdot L''$, we may assume $L' \subset L''$. Then we calculate

$$
\begin{aligned}
&\frac{1}{[L'':K]} \sum_{w \in M(L'')} [L''_w : K_{w|_K}] \lambda_{\mathcal{Y},L''}(x,w) \\
&= \frac{1}{[L'':K]} \sum_{v \in M(L')} \sum_{\substack{w \in M(L'') \\ w|_{L'} = v}} [L''_w : L'_v][L'_v : K_{v|_K}] \lambda_{\mathcal{Y},L''}(x,w) \\
&= \frac{1}{[L'':K]} \sum_{v \in M(L')} \sum_{\substack{w \in M(L'') \\ w|_{L'} = v}} [L''_w : L'_v][L'_v : K_{v|_K}] \lambda_{\mathcal{Y},L'}(x,v) \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{since } x \in (X\backslash Y)(L') \\
&= \frac{[L'':L']}{[L'':K]} \sum_{v \in M(L')} [L'_v : K_{v|_K}] \lambda_{\mathcal{Y},L'}(x,v) \qquad \text{by well-behavedness} \\
&= \frac{1}{[L':K]} \sum_{v \in M(L')} [L'_v : K_{v|_K}] \lambda_{\mathcal{Y},L'}(x,v)
\end{aligned}
$$

and we are done. □

Construction 7.3 (Global height function associated with a closed subscheme)**.** Let X be a projective scheme over $\overline{K}$. Let $Y \subset X$ be a closed subscheme such that $Y \cap \mathrm{Ass}(X) = \varnothing$. Take an intermediate field $K \subset L \subset \overline{K}$ such that

- $[L : K] < \infty$;
- there is a projective scheme X_L and a closed subscheme $Y_L \subset X_L$ such that $(X_L)_{\overline{K}} \simeq X$ as $\overline{K}$-schemes and $(Y_L)_{\overline{K}} \simeq Y$ via this isomorphism:

$$
\begin{tikzcd}
(X_L)_{\overline{K}} \arrow[r, "\sim"] & X \\
(Y_L)_{\overline{K}} \arrow[r, "\sim"] \arrow[u, hook] & Y \arrow[u, hook]
\end{tikzcd}
$$

We identify $(X_L)_{\overline{K}}$ with X by this fixed isomorphism. Note that $Y_L \cap \mathrm{Ass}(X_L) = \varnothing$. Take a presentation $\mathcal{Y}$ of Y_L. The class of $h_{\mathcal{Y}}$ in the set of maps from $(X\backslash Y)(\overline{K})$ to $\mathbb{R}$ modulo bounded functions is called the global height function associated with Y. Each element of the equivalence class is called a global height function associated with Y.

Claim 7.4. *This is well-defined, that is, if L' and $\mathcal{Y}'$ are another such choices, then $h_{\mathcal{Y}} - h_{\mathcal{Y}'}$ is a bounded function.*

Proof. Let L'' be a sufficiently large finite extension of the composite field $L \cdot L'$ such that the fixed isomorphism $(X_L)_{\overline{K}} \simeq X \simeq (X_{L'})_{\overline{K}}$ is defined over L''. Then $\mathcal{Y}_{L''}$ and $\mathcal{Y}'_{L''}$ are presentations of $(Y_L)_{L''} \simeq (Y_{L'})_{L''}$ (the isomorphism via the fixed isomorphism $(X_L)_{L''} \simeq (X_{L'})_{L''}$). Also, we have $h_{\mathcal{Y}_{L''}} = h_{\mathcal{Y}}$ and $h_{\mathcal{Y}'_{L''}} = h_{\mathcal{Y}'}$. Thus we may assume $L = L'$ and $\mathcal{Y}$ and $\mathcal{Y}'$ are two presentations of $Y_L \subset X_L$.

By Proposition 4.25, there is an M_K-constant γ such that

$$|\lambda_{\mathcal{Y}}(x, v) - \lambda_{\mathcal{Y}'}(x, v)| \leqslant \gamma(v)$$

for $(x, v) \in (X\backslash Y)(\overline{K}) \times M(\overline{K})$.

Take any $x \in (X\backslash Y)(\overline{K}) \times M(\overline{K})$. Take any intermediate field $L \subset L' \subset \overline{K}$ such that $[L' : L] < \infty$ and $x \in (X_L\backslash Y_L)(L')$. Then

$$\begin{aligned} &h_{\mathcal{Y}}(x) - h_{\mathcal{Y}'}(x) \\ &= \frac{1}{[L' : K]} \sum_{v \in M(L')} [L'_v : K_{v|_K}]\Big(\lambda_{\mathcal{Y},L'}(x, v) - \lambda_{\mathcal{Y}',L'}(x, v)\Big). \end{aligned}$$

For $w \in M(\overline{K})$, set $v = w|_{L'}$ and $v_0 = w|_K (= v|_K)$. Then by the definitions of local heights, we have

$$\begin{aligned} \lambda_{\mathcal{Y},L'}(x, v) &= \lambda_{\mathcal{Y}}(x, w); \\ \lambda_{\mathcal{Y}',L'}(x, v) &= \lambda_{\mathcal{Y}'}(x, w) \end{aligned}$$

and therefore

$$|\lambda_{\mathcal{Y},L'}(x, v) - \lambda_{\mathcal{Y}',L'}(x, v)| \leqslant \gamma(v_0).$$

Thus

$$\begin{aligned} |h_{\mathcal{Y}}(x) - h_{\mathcal{Y}'}(x)| &\leqslant \frac{1}{[L' : K]} \sum_{v \in M(L')} [L'_v : K_{v|_K}]\gamma(v|_K) \\ &= \frac{1}{[L' : K]} \sum_{v_0 \in M_K} \gamma(v_0) \sum_{\substack{v \in M(L') \\ v|_K = v_0}} [L'_v : K_{v_0}] \end{aligned}$$

$$= \sum_{v_0 \in M_K} \gamma(v_0)$$

and we are done. □

Definition 7.5. Notation as in Construction 7.3. The equivalence class of $h_{\mathcal{Y}}$ modulo bounded functions is denoted by h_Y and called the global height function associated with Y. Each representative function is usually denoted by h_Y too and called a global height function associated with Y.

Proposition 7.6 (Basic properties of global height functions)**.** *Let X be a projective scheme over $\overline{K}$. Let $Y, Z \subset X$ be closed subschemes such that $Y \cap \mathrm{Ass}(X) = Z \cap \mathrm{Ass}(X) = \varnothing$. Then the following hold.*

(1) $h_{Y \cap Z} \leqslant \min\{h_Y, h_Z\} + O(1)$ *on* $(X \backslash Y \cap Z)(\overline{K})$.

(2) $h_{Y+Z} = h_Y + h_Z + O(1)$ *on* $(X \backslash (Y \cup Z))(\overline{K})$.

(3) *If* $Y \subset Z$, *then* $h_Y \leqslant h_Z + O(1)$ *on* $(X \backslash Y)(\overline{K})$.

(4) $\max\{h_Y, h_Z\} \leqslant h_{Y \cup Z} + O(1) \leqslant h_Y + h_Z + O(1)$ *on* $(X \backslash (Y \cup Z))(\overline{K})$.

(5) *If* $\mathrm{Supp}\, Y \subset \mathrm{Supp}\, Z$, *then there is a positive constant* $c > 0$ *such that*

$$h_Y \leqslant c h_Z + O(1)$$

on $(X \backslash Y)(\overline{K})$.

(6) *Let* X' *be a projective scheme over* $\overline{K}$ *and* $\varphi \colon X' \longrightarrow X$ *be a morphism over* $\overline{K}$. *Suppose* $\varphi(\mathrm{Ass}(X')) \cap Y = \varnothing$. *Then*

$$h_Y \circ \varphi = h_{\varphi^{-1}(Y)} + O(1)$$

on $(X' \backslash \varphi^{-1}(Y))(\overline{K})$.

Proof. These follow from the definition of global height, Proposition 4.21, and Theorem 4.33. □

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